An Introduction to Interpolation Theory

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Contents

Introduction

Let X, Y be two real or complex Banach spaces. By $X = Y$ we mean that X and Y have the same elements and equivalent norms. By $Y \subset X$ we mean that Y is continuously embedded in X.

The couple of Banach spaces (X, Y) is said to be an *interpolation couple* if both X and Y are continuously embedded in a Hausdorff topological vector space $\mathcal{V}$. In this case the intersection $X \cap Y$ is a linear subspace of $\mathcal{V}$, and it is a Banach space under the norm

$$\|v\|_{X\cap Y} = \max\{\|v\|_X,\ \|v\|_Y\}.$$

Also the sum $X + Y = \{x + y :\ x \in X,\ y \in Y\}$ is a linear subspace of $\mathcal{V}$. It is endowed with the norm

$$\|v\|_{X+Y} = \inf_{x\in X,\, y\in Y,\, x+y=v} \|x\|_X + \|y\|_Y.$$

As easily seen, $X + Y$ is isometric to the quotient space $(X \times Y)/D$, where $D = \{(x, -x) : x \in X \cap Y\}$. Since $\mathcal{V}$ is a Hausdorff space, then D is closed, and $X + Y$ is a Banach space. Moreover, $\|x\|_X \leq \|x\|_{X+Y}$ and $\|y\|_Y \leq \|y\|_{X+Y}$ for all $x \in X$, $y \in Y$, so that both X and Y are continuously embedded in $X + Y$.

The space $\mathcal{V}$ is used only to guarantee that $X + Y$ is a Banach space. It will disappear from the general theory.

If (X, Y) is an interpolation couple, an *intermediate space* is any Banach space E such that

$$X \cap Y \subset E \subset X + Y.$$

An *interpolation space* between X and Y is any intermediate space such that for every $T \in L(X) \cap L(Y)$ (that is, for every $T \in L(X + Y)$ whose restriction to X belongs to $L(X)$ and whose restriction to Y belongs to $L(Y)$), the restriction of T to E belongs to $L(E)$. We could also require that there is a constant independent of T such that $\|T\|_{L(E)} \leq C(\|T\|_{L(X)} + \|T\|_{L(Y)})$, but often this property is neglected.

The general interpolation theory is not devoted to characterize all the interpolation spaces between X and Y but rather to construct suitable families of interpolation spaces and to study their properties. The most known and useful families of interpolation spaces are the *real interpolation spaces* which will be treated in chapter 1, and the *complex interpolation spaces* which will be treated in chapter 2.

Interpolation theory has a wide range of applications. We shall emphasize applications to partial differential operators and partial differential equations, referring to [36], [12] for applications to other fields. In particular we shall give self-contained proofs of optimal regularity results in Hölder and in fractional Sobolev spaces for linear elliptic and parabolic differential equations.

The domains of powers of positive operators in Banach spaces are not interpolation spaces in general. However in some interesting cases they coincide with suitable complex interpolation spaces. In any case the theory of powers of positive operators is very close to interpolation theory, and there are important connections between them. Therefore in chapter 4 we give an elementary treatment of the powers of positive operators, with particular attention to the imaginary powers.

Chapter 1

Real interpolation

Let (X, Y) be a real or complex interpolation couple.

If I is any interval contained in $(0, +\infty)$, $L^p_*(I)$ is the Lebesgue space L^p with respect to the measure dt/t in I. In particular, $L^\infty_*(I) = L^\infty(I)$. See Appendix, §2.

1.1 The K-method

Definition 1.1.1 *For every $x \in X + Y$ and $t > 0$, set*

$$K(t, x, X, Y) = \inf_{x=a+b,\, a\in X,\, b\in Y} (\|a\|_X + t\|b\|_Y)\,. \tag{1.1}$$

If there is no danger of confusion, we shall write $K(t, x)$ $K(t, x, X, Y)$ instead of $K(t, x, X, Y)$.

Note that $K(1, x) = \|x\|_{X+Y}$, and for every $t > 0$ $K(t, \cdot)$ is a norm in $X+Y$, equivalent to the norm of $X + Y$. Now we define a family of Banach spaces by means of the function K.

Definition 1.1.2 *Let $0 < \theta < 1$, $1 \le p \le \infty$, and set*

$$\begin{cases} (X, Y)_{\theta,p} = \{x \in X + Y : t \mapsto t^{-\theta} K(t, x, X, Y) \in L^p_*(0, +\infty)\}, \\ \\ \|x\|_{(X,Y)_{\theta,p}} = \|t^{-\theta} K(t, x, X, Y)\|_{L^p_*(0,\infty)}; \end{cases} \tag{1.2}$$

$$\begin{aligned} (X, Y)_\theta \;&=\; \{x \in X + Y : \lim_{t\to 0^+} t^{-\theta} K(t, x, X, Y) \\ &\qquad = \lim_{t\to +\infty} t^{-\theta} K(t, x, X, Y) = 0\}. \end{aligned} \tag{1.3}$$

Such spaces are called real interpolation spaces.

Since $t \mapsto K(t, x)$ is continuous in $(0, \infty)$ for each $x \in X+Y$, then $(X, Y)_\theta \subset (X, Y)_{\theta,\infty}$. The spaces $(X, Y)_\theta$ are also called *continuous interpolation spaces.*

The mapping $x \mapsto \|x\|_{(X,Y)_{\theta,p}}$ is easily seen to be a norm in $(X, Y)_{\theta,p}$. If no confusion may arise, we shall write $\|x\|_{\theta,p}$ instead of $\|x\|_{(X,Y)_{\theta,p}}$.

Note that $K(t, x, X, Y) = tK(t^{-1}, x, Y, X)$ for each $t > 0$. By the transformation $\tau = t^{-1}$, which preserves $L^p_*(0, \infty)$, we get

$$(X, Y)_{\theta,p} = (Y, X)_{1-\theta,p}, \;\; 0 < \theta < 1, \; 1 \le p \le \infty, \tag{1.4}$$

and

$$(X,Y)_\theta = (Y,X)_{1-\theta}. \tag{1.5}$$

So, pay attention to the order!

Let us consider some particular cases.

(a) Let $X = Y$. Then $X + Y = X$, and $K(t,x) \le \min\{t,1\}\|x\|$. Therefore

$$X \subset (X,X)_{\theta,p}, \;\; 0<\theta<1, \; 1 \le p \le \infty.$$

In the next proposition we will see that for any interpolation couple we have $(X,Y)_{\theta,p} \subset X+Y$. So, if $X = Y$ then $(X,X)_{\theta,p} = X$.

(b) If $X \cap Y = \{0\}$, then for each $x \in X+Y$ there are a unique $a \in X$ and a unique $b \in Y$ such that $x = a+b$, hence $K(t,x) = \|a\|_X + t\|b\|_Y$ and $t \mapsto t^{-\theta}K(t,x)$ does not belong to any $L^p_*(0,\infty)$, unless $x = 0$. Therefore, $(X,Y)_{\theta,p} = (X,Y)_\theta = \{0\}$ for every $p \in [1,\infty]$, $\theta \in (0,1)$.

(c) In the important case where $Y \subset X$ we have $K(t,x) \le \|x\|_X$ for every $x \in X$, so that $t \mapsto t^{-\theta}K(t,x) \in L^p_*(a,\infty)$ for all $a > 0$, and $\lim_{t\to+\infty} t^{-\theta}K(t,x) = 0$. Therefore, only the behavior near $t = 0$ of $t^{-\theta}K(t,x)$ plays a role in the definition of $(X,Y)_{\theta,p}$ and of $(X,Y)_\theta$. Indeed, one could replace the halfline $(0,+\infty)$ by any interval $(0,a)$ in definition 1.1.2, obtaining equivalent norms.

The inclusion properties of the real interpolation spaces are stated below.

Proposition 1.1.3 *For $0<\theta<1$, $1 \le p_1 \le p_2 \le \infty$ we have*

$$X \cap Y \subset (X,Y)_{\theta,p_1} \subset (X,Y)_{\theta,p_2} \subset (X,Y)_\theta \subset (X,Y)_{\theta,\infty} \subset X+Y. \tag{1.6}$$

Moreover,

$$(X,Y)_{\theta,\infty} \subset \overline{X} \cap \overline{Y},$$

where $\overline{X}$, $\overline{Y}$ are the closures of X, Y in $X+Y$.

Proof. Let us show that $(X,Y)_{\theta,\infty}$ is contained in $\overline{X}\cap\overline{Y}$ and it is continuously embedded in $X+Y$. For $x \in (X,Y)_{\theta,\infty}$ we have

$$K(t,x) = \inf_{x=a+b} \|a\|_X + t\|b\|_Y \le t^\theta \|x\|_{\theta,\infty}, \;\; t>0,$$

so that for every $n \in \mathbb{N}$ (taking $t = 1/n$) there are $a_n \in X$, $b_n \in Y$ such that $x = a_n + b_n$, and

$$\|a_n\|_X + \frac{1}{n}\|b_n\|_Y \le \frac{2}{n^\theta}\|x\|_{\theta,\infty}.$$

In particular, $\|x - b_n\|_{X+Y} = \|a_n\|_{X+Y} \le \|a_n\|_X \le 2\|x\|_{\theta,\infty} n^{-\theta}$, so that the sequence $\{b_n\}$ goes to x in $X+Y$ as $n \to \infty$. This implies that $(X,Y)_{\theta,\infty}$ is contained in $\overline{Y}$. Arguing similarly (i.e., replacing $1/n$ by n and letting $n \to \infty$), or else recalling that $(X,Y)_{\theta,\infty} = (Y,X)_{1-\theta,\infty}$) we see that $(X,Y)_{\theta,\infty}$ is contained also in $\overline{X}$. Moreover, by definition $\|x\|_{X+Y} = K(1,x)$. Therefore

$$\|x\|_{X+Y} = K(1,x) \le \|x\|_{\theta,\infty}, \;\; \forall x \in (X,Y)_{\theta,\infty},$$

so that $(X,Y)_{\theta,\infty}$ is continuously embedded in $X+Y$.

The inclusion $(X,Y)_\theta \subset (X,Y)_{\theta,\infty}$ is obvious because $K(\cdot,x)$ is continuous (see exercise 1, §1.1.1) so that $t^{-\theta}K(t,x)$ is bounded in every interval $[a,b]$ with $0<a<b$.

Let us show that $(X,Y)_{\theta,p}$ is contained in $(X,Y)_\theta$ and it is continuously embedded in $(X,Y)_{\theta,\infty}$ for $p<\infty$. For each $x\in(X,Y)_{\theta,p}$ and $t>0$, recalling that $K(\cdot,x)$ is increasing we get

$$\begin{aligned} t^{-\theta}K(t,x) &= (\theta p)^{1/p}\left(\int_t^{+\infty} s^{-\theta p-1}ds\right)^{1/p} K(t,x) \\ &\le (\theta p)^{1/p}\left(\int_t^{+\infty} s^{-\theta p-1}K(s,x)^p ds\right)^{1/p}, \quad t>0. \end{aligned} \tag{1.7}$$

The right hand side is bounded by $(\theta p)^{1/p}\|x\|_{\theta,p}$. Therefore $x\in(X,Y)_{\theta,\infty}$, and $\|x\|_{\theta,\infty}\le(\theta p)^{1/p}\|x\|_{\theta,p}$. Changing θ with $1-\theta$ and X with Y we obtain $\|x\|_{\theta,\infty}\le((1-\theta)p)^{1/p}\|x\|_{\theta,p}$. Therefore,

$$\|x\|_{\theta,\infty}\le[\min\{\theta,1-\theta\}p]^{1/p}\|x\|_{\theta,p}. \tag{1.8}$$

Moreover letting $t\to\infty$ we get $\lim_{t\to\infty}t^{-\theta}K(t,x)=0$. To prove that $x\in(X,Y)_\theta$ we need also that $\lim_{t\to 0}t^{-\theta}K(t,x)=0$. This can be seen as follows: since $(X,Y)_{\theta,p}=(Y,X)_{1-\theta,p}$ then

$$0=\lim_{t\to+\infty}t^{-(1-\theta)}K(t,x,Y,X)=\lim_{t\to+\infty}t^{\theta}K(t^{-1},x,X,Y)=\lim_{\tau\to 0^+}\tau^{-\theta}K(\tau,x,X,Y).$$

Let us prove that $(X,Y)_{\theta,p_1}\subset(X,Y)_{\theta,p_2}$ for $p_1<p_2<+\infty$. For $x\in(X,Y)_{\theta,p_1}$ we have

$$\begin{aligned} \|x\|_{\theta,p_2} &= \left(\int_0^{+\infty} t^{-\theta p_2}K(t,x)^{p_2}\frac{dt}{t}\right)^{1/p_2} \\ &= \left(\int_0^{+\infty} t^{-\theta p_1}K(t,x)^{p_1}(t^{-\theta}K(t,x))^{p_2-p_1}\frac{dt}{t}\right)^{1/p_2} \\ &\le \left(\int_0^{+\infty} t^{-\theta p_1}K(t,x)^{p_1}\frac{dt}{t}\right)^{1/p_2}\left(\sup_{t>0}t^{-\theta}K(t,x)\right)^{(p_2-p_1)/p_2} \\ &= (\|x\|_{\theta,p_1})^{p_1/p_2}(\|x\|_{\theta,\infty})^{1-p_1/p_2}, \end{aligned}$$

and using (1.8) we find

$$\|x\|_{\theta,p_2}\le[\min\{\theta,1-\theta\}p_1]^{1/p_1-1/p_2}\|x\|_{\theta,p_1}. \tag{1.9}$$

Finally, from the inequality $K(t,x)\le\min\{1,t\}\ \|x\|_{X\cap Y}$ for every $x\in X\cap Y$ it follows immediately that $X\cap Y$ is continuously embedded in $(X,Y)_{\theta,p}$ for $0<\theta<1$, $1\le p\le\infty$.

The statement is so completely proved. □

The first part of the proof of proposition 1.1.3 shows the connection between interpolation theory and approximation theory. Indeed, the sequence b_n in the proof consists of elements of Y and converges to x in $X+Y$. The rate of convergence of b_n and the rate of blowing up of $\|b_n\|_Y$ are described precisely by the fact that $x\in(X,Y)_{\theta,\infty}$ (or $x\in(X,Y)_{\theta,p}\subset(X,Y)_{\theta,\infty}$). In particular, if $x\in(X,Y)_{\theta,\infty}$ then $\|x-b_n\|_{X+Y}\le$ const. $n^{-\theta}$, and $\|b_n\|_Y\le$ const. $n^{1-\theta}$.

If $Y\subset X$ other embeddings hold.

Proposition 1.1.4 *If $Y \subset X$, for $0 < \theta_1 < \theta_2 < 1$ we have*

$$(X,Y)_{\theta_2,\infty} \subset (X,Y)_{\theta_1,1}. \tag{1.10}$$

Therefore, $(X,Y)_{\theta_2,p} \subset (X,Y)_{\theta_1,q}$ for every p, $q \in [1,\infty]$.

Proof. For $x \in (X,Y)_{\theta_2,\infty}$ we have, using the inequalities $K(t,x) \le \|x\|_X$ for $t \ge 1$ and $K(t,x) \le t^{\theta_2}\|x\|_{\theta_2,\infty}$ for $0 < t \le 1$,

$$\begin{aligned}\|x\|_{\theta_1,1} &= \int_0^1 t^{-\theta_1-1}K(t,x)dt + \int_1^{+\infty} t^{-\theta_1-1}K(t,x)dt \\ &\le \int_0^1 t^{-\theta_1-1}\|x\|_{\theta_2,\infty}t^{\theta_2}dt + \int_1^{+\infty} t^{-\theta_1-1}\|x\|_X dt \qquad (1.11)\\ &\le \frac{1}{\theta_2-\theta_1}\|x\|_{\theta_2,\infty} + \frac{1}{\theta_1}\|x\|_X,\end{aligned}$$

and the statement follows since $(X,Y)_{\theta_2,\infty} \subset X+Y = X$ because $Y \subset X$. □

Note that (1.10) is not true in general. See next example 1.1.10.

Proposition 1.1.5 *For all $\theta \in (0,1)$, $p \in [1,\infty]$, $(X,Y)_{\theta,p}$ is a Banach space. For all $\theta \in (0,1)$, $(X,Y)_\theta$ is a Banach space, endowed with the norm of $(X,Y)_{\theta,\infty}$.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $(X,Y)_{\theta,p}$. By the continuous embedding $(X,Y)_{\theta,p} \subset X+Y$, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $X+Y$ too, so that it converges to an element $x \in X+Y$.

Let us estimate $\|x_n - x\|_{\theta,p}$. Fix $\varepsilon > 0$, and let $\|x_n - x_m\|_{\theta,p} \le \varepsilon$ for n, $m \ge n_\varepsilon$. Since $y \mapsto K(t,y)$ is a norm in $X+Y$, for every n, $m \in \mathbb{N}$ and $t > 0$ we have $K(t,x_n - x) \le K(t,x_n - x_m) + K(t,x_m - x)$, so that

$$t^{-\theta}K(t,x_n - x) \le t^{-\theta}K(t,x_n - x_m) + t^{-\theta}\max\{t,1\}\|x_m - x\|_{X+Y}. \tag{1.12}$$

Let $p = \infty$. Then for every $t > 0$ and n, $m \ge n_\varepsilon$

$$t^{-\theta}K(t,x_n - x) \le \varepsilon + t^{-\theta}\max\{t,1\}\|x_m - x\|_{X+Y}.$$

Letting $m \to +\infty$, we find $t^{-\theta}K(t,x_n - x) \le \varepsilon$ for every $t > 0$. This implies that $x \in (X,Y)_{\theta,\infty}$ and that $x_n \to x$ in $(X,Y)_{\theta,\infty}$. Therefore $(X,Y)_{\theta,\infty}$ is complete.

It is easy to see that $(X,Y)_\theta$ is a closed subspace of $(X,Y)_{\theta,\infty}$. Since $(X,Y)_{\theta,\infty}$ is complete, then also $(X,Y)_\theta$ is complete.

Let now $p < \infty$. Then

$$\|x_n - x\|_{\theta,p} = \lim_{\delta\to 0}\left(\int_\delta^{1/\delta} t^{-\theta p-1}K(t,x_n - x)^p dt\right)^{1/p}.$$

Due again to (1.12), for every $\delta \in (0,1)$ we get, for n, $m \ge n_\varepsilon$,

$$\begin{aligned}&\left(\int_\delta^{1/\delta} t^{-\theta p-1}K(x_n - x)^p dt\right)^{1/p} \le \|x_n - x_m\|_{\theta,p} \\ &+\|x_m - x\|_{X+Y}\left(\int_\delta^{1/\delta} t^{-\theta p-1}\max\{t,1\}dt\right)^{1/p} \le \varepsilon + C(\delta,p)\|x_m - x\|_{X+Y}.\end{aligned}$$

Letting first $m \to \infty$ and then $\delta \to 0$ we get $x \in (X,Y)_{\theta,p}$ and $x_n \to x$ in $(X,Y)_{\theta,p}$. So, $(X,Y)_{\theta,p}$ is complete. □

The spaces $(X,Y)_{\theta,p}$ and $(X,Y)_\theta$ are interpolation spaces, as a consequence of the following important theorem.

Theorem 1.1.6 *Let (X_1,Y_1), (X_2,Y_2) be interpolation couples. If $T \in L(X_1,X_2) \cap L(Y_1,Y_2)$, then $T \in L((X_1,Y_1)_{\theta,p},(X_2,Y_2)_{\theta,p}) \cap L((X_1,Y_1)_\theta, (X_2,Y_2)_\theta)$ for every $\theta \in (0,1)$ and $p \in [1,\infty]$. Moreover,*

$$\|T\|_{L((X_1,Y_1)_{\theta,p},(X_2,Y_2)_{\theta,p})} \leq (\|T\|_{L(X_1,X_2)})^{1-\theta}(\|T\|_{L(Y_1,Y_2)})^{\theta}. \tag{1.13}$$

Proof. If T is the null operator, there is nothing to prove. If $T \neq 0$, either $\|T\|_{L(X_1,X_2)} \neq 0$ or $\|T\|_{L(Y_1,Y_2)} \neq 0$. Assume that $\|T\|_{L(X_1,X_2)} \neq 0$. Let $x \in (X_1,Y_1)_{\theta,p}$: then for every $a \in X_1$, $b \in Y_1$ such that $x = a+b$ and for every $t > 0$ we have

$$\|Ta\|_{X_2} + t\|Tb\|_{Y_2} \leq \|T\|_{L(X_1,X_2)} \left(\|a\|_{X_1} + t\frac{\|T\|_{L(Y_1,Y_2)}}{\|T\|_{L(X_1,X_2)}}\|b\|_{Y_1} \right),$$

so that, taking the infimum over all a, b as above, we get

$$K(t,Tx,X_2,Y_2) \leq \|T\|_{L(X_1,X_2)} K\left(t\frac{\|T\|_{L(Y_1,Y_2)}}{\|T\|_{L(X_1,X_2)}}, x, X_1, Y_1 \right). \tag{1.14}$$

Setting $s = t\frac{\|T\|_{L(Y_1,Y_2)}}{\|T\|_{L(X_1,X_2)}}$ we get $Tx \in (X_2,Y_2)_{\theta,p}$, and

$$\|Tx\|_{(X_2,Y_2)_{\theta,p}} \leq \|T\|_{L(X_1,X_2)} \left(\frac{\|T\|_{L(Y_1,Y_2)}}{\|T\|_{L(X_1,X_2)}} \right)^{\theta} \|x\|_{(X_1,Y_1)_{\theta,p}},$$

and (1.13) follows. From (1.14) it follows also that

$$\lim_{t\to 0} t^{-\theta}K(t,x,X_1,Y_1) = \lim_{t\to\infty} t^{-\theta}K(t,x,X_1,Y_1) = 0 \Longrightarrow$$

$$\Longrightarrow \lim_{t\to 0} t^{-\theta}K(t,Tx,X_2,Y_2) = \lim_{t\to\infty} t^{-\theta}K(t,Tx,X_2,Y_2) = 0,$$

that is, T maps $(X_1,Y_1)_\theta$ into $(X_2,Y_2)_\theta$.

In the case where $\|T\|_{L(X_1,X_2)} = 0$ we get the result either replacing everywhere $\|T\|_{L(X_1,X_2)}$ by $\varepsilon > 0$ and then letting $\varepsilon \to 0$, or else replacing X_i by Y_i for $i = 1,2$ and θ by $1-\theta$ (see (1.4) and (1.5)). □

Taking $X_1 = X_2 = X$, $Y_1 = Y_2 = Y$, it follows that $(X,Y)_{\theta,p}$ and $(X,Y)_\theta$ are interpolation spaces. Another important consequence is the next corollary.

Corollary 1.1.7 *Let (X,Y) be an interpolation couple. For $0 < \theta < 1$, $1 \leq p \leq \infty$ there is $c(\theta,p)$ such that*

$$\|y\|_{(X,Y)_{\theta,p}} \leq c(\theta,p)\|y\|_X^{1-\theta}\|y\|_Y^{\theta} \ \ \forall y \in X \cap Y. \tag{1.15}$$

Proof. Set $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, according to the fact that X, Y are real or complex Banach spaces. Let $y \in X \cap Y$, and define T by $T(\lambda) = \lambda y$ for each $\lambda \in \mathbb{K}$. Then $\|T\|_{L(\mathbb{K},X)} = \|y\|_X$, $\|T\|_{L(\mathbb{K},Y)} = \|y\|_Y$, and $\|T\|_{L(\mathbb{K},(X,Y)_{\theta,p})} = \|y\|_{(X,Y)_{\theta,p}}$. The statement follows now taking $X_1 = Y_1 = \mathbb{K}$ and $X_2 = X$, $Y_2 = Y$ in theorem 1.1.6, and recalling that $(\mathbb{K},\mathbb{K})_{\theta,p} = \mathbb{K}$.

Another more direct proof is the following: for $y \in X \cap Y \setminus \{0\}$, we have $K(t,y) \leq \min\{\|y\|_X, t\|y\|_Y\}$, so that

$$t \leq \frac{\|y\|_X}{\|y\|_Y} \Longrightarrow K(t,y) \leq t\|y\|_Y \Longrightarrow t^{-\theta}K(t,y) \leq t^{1-\theta}\|y\|_Y \leq \|y\|_X^{1-\theta}\|y\|_Y^{\theta},$$

and

$$t \geq \frac{\|y\|_X}{\|y\|_Y} \Longrightarrow K(t,y) \leq \|y\|_X \Longrightarrow t^{-\theta}K(t,y) \leq \left(\frac{\|y\|_Y}{\|y\|_X}\right)^{\theta}\|y\|_X = \|y\|_X^{1-\theta}\|y\|_Y^{\theta}.$$

Therefore $\|y\|_{(X,Y)_{\theta,\infty}} = \sup_{t>0} t^{-\theta}K(t,y) \leq \|y\|_X^{1-\theta}\|y\|_Y^{\theta}$, and the statement follows for $p = +\infty$ with constant $c(\theta,\infty) = 1$. For $p < +\infty$ we already know that $(X,Y)_{\theta,p}$ is continuously embedded in $(X,Y)_{\theta,\infty}$, and the proof is complete. □

1.1.1 Examples

Let us see some basic examples. $C_b(\mathbb{R}^n)$ is the space of the bounded continuous functions in $\mathbb{R}^n$, endowed with the sup norm $\|\cdot\|_\infty$; $C_b^1(\mathbb{R}^n)$ is the subset of the continuously differentiable functions with bounded derivatives, endowed with the norm $\|f\|_\infty + \sum_{i=1}^n \|D_i f\|_\infty$. For $\theta \in (0,1)$, $C_b^\theta(\mathbb{R}^n)$ is the set of the bounded and uniformly Hölder continuous functions, endowed with the norm

$$\|f\|_{C_b^\theta} = \|f\|_\infty + [f]_{C^\theta} = \|f\|_\infty + \sup_{x\neq y}\frac{|f(x)-f(y)|}{|x-y|^\theta}.$$

For $\theta \in (0,1)$, $p \in [1,\infty)$, $W^{\theta,p}(\mathbb{R}^n)$ is the space of all $f \in L^p(\mathbb{R}^n)$ such that

$$[f]_{W^{\theta,p}} = \left(\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|f(x)-f(y)|^p}{|x-y|^{\theta p+n}}dx\,dy\right)^{1/p} < \infty.$$

It is endowed with the norm $\|\cdot\|_{L^p} + [\cdot]_{W^{\theta,p}}$.

Example 1.1.8 *For $0 < \theta < 1$, $1 \leq p < \infty$ we have*

$$(C_b(\mathbb{R}^n), C_b^1(\mathbb{R}^n))_{\theta,\infty} = C_b^\theta(\mathbb{R}^n), \tag{1.16}$$

$$(L^p(\mathbb{R}^n), W^{1,p}(\mathbb{R}^n))_{\theta,p} = W^{\theta,p}(\mathbb{R}^n), \tag{1.17}$$

with equivalence of the respective norms.

Proof. Let us prove the first statement. Let $f \in (C_b(\mathbb{R}^n), C_b^1(\mathbb{R}^n))_{\theta,\infty}$. Since $(X,Y)_{\theta,\infty} \subset X + Y$, we have $\|f\|_\infty \leq \text{const.}\,\|f\|_{\theta,\infty}$. In our case the constant is 1, because for every decomposition $f = a + b$, with $a \in C_b(\mathbb{R}^n)$, $b \in C_b^1(\mathbb{R}^n)$ we have $\|f\|_\infty \leq \|a\|_\infty + \|b\|_\infty$, so that

$$\|f\|_\infty \leq K(1, f, C_b(\mathbb{R}^n), C_b^1(\mathbb{R}^n)) \leq \|f\|_{\theta,\infty}.$$

Moreover for $x \neq y$ and again for every decomposition $f = a + b$, with $a \in C_b(\mathbb{R}^n)$, $b \in C_b^1(\mathbb{R}^n)$, we have

$$|f(x) - f(y)| \leq |a(x) - a(y)| + |b(x) - b(y)| \leq 2\|a\|_\infty + \|b\|_{C^1}|x - y|,$$

so that, taking the infimum over all the decompositions,

$$|f(x) - f(y)| \leq 2K(|x - y|, f, C_b(\mathbb{R}^n), C_b^1(\mathbb{R}^n)) \leq 2|x - y|^\theta \|f\|_{\theta,\infty}.$$

Therefore f is θ-Hölder continuous and $\|f\|_{C^\theta} = \|f\|_\infty + [f]_{C^\theta} \leq 3\|f\|_{\theta,\infty}$.

Conversely, let $f \in C_b^\theta(\mathbb{R}^n)$. Let $\varphi \in C^\infty(\mathbb{R}^n)$ be a nonnegative function with support in the unit ball, such that $\int_{\mathbb{R}^n} \varphi(x)dx = 1$. For every $t > 0$ set

$$b_t(x) = \frac{1}{t^n} \int_{\mathbb{R}^n} f(y)\varphi\left(\frac{x - y}{t}\right)dy, \quad a_t(x) = f(x) - b_t(x), \quad x \in \mathbb{R}^n. \tag{1.18}$$

Then

$$a_t(x) = \frac{1}{t^n} \int_{\mathbb{R}^n} (f(x) - f(x - y))\varphi\left(\frac{y}{t}\right)dy$$

so that

$$\|a_t\|_\infty \leq [f]_{C^\theta} \frac{1}{t^n} \int_{\mathbb{R}^n} |y|^\theta \varphi(y/t)dy = t^\theta [f]_{C^\theta} \int_{\mathbb{R}^n} |w|^\theta \varphi(w)dw.$$

Moreover, $\|b_t\|_\infty \leq \|f\|_\infty$, and

$$D_i b_t(x) = \frac{1}{t^{n+1}} \int_{\mathbb{R}^n} f(y) D_i\varphi\left(\frac{x - y}{t}\right)dy.$$

Since $\int_{\mathbb{R}^n} D_i\varphi((x - y)/t)dy = 0$, we get

$$D_i b_t(x) = \frac{1}{t^{n+1}} \int_{\mathbb{R}^n} (f(x - y) - f(x)) D_i\varphi\left(\frac{y}{t}\right)dy, \tag{1.19}$$

which implies

$$\|D_i b_t\|_\infty \leq t^{\theta-1}[f]_{C^\theta} \int_{\mathbb{R}^n} |w|^\theta |D_i\varphi(w)|dw.$$

Therefore,

$$t^{-\theta} K(t, f) \leq t^{-\theta}(\|a_t\|_\infty + t\|b_t\|_{C^1}) \leq C\|f\|_{C^\theta}, \quad 0 < t \leq 1.$$

For $t \geq 1$ (see remark (c) after definition 1.1.1), we can take $a_t = f$, $b_t = 0$ which implies

$$t^{-\theta} K(t, f) \leq t^{-\theta}\|f\|_\infty \leq \|f\|_\infty, \quad t \geq 1.$$

The embedding $C_b^\theta(\mathbb{R}^n) \subset (C_b(\mathbb{R}^n), C_b^1(\mathbb{R}^n))_{\theta,\infty}$ follows.

The proof of the second statement is similar. We recall that for every $b \in W^{1,p}(\mathbb{R}^n)$ and $h \in \mathbb{R}^n \setminus \{0\}$ we have (see e.g. [8])

$$\left(\int_{\mathbb{R}^n} \left(\frac{|b(x + h) - b(x)|}{|h|}\right)^p dx\right)^{1/p} \leq \|Db\|_{L^p}.$$

For every $f \in (L^p(\mathbb{R}^n), W^{1,p}(\mathbb{R}^n))_{\theta,p}$ and $h \in \mathbb{R}^n$ let $a = a(h) \in L^p(\mathbb{R}^n)$, $b = b(h) \in W^{1,p}(\mathbb{R}^n)$ be such that $f = a + b$, and

$$\|a\|_{L^p} + |h| \, \|b\|_{W^{1,p}} \leq 2K(|h|, f).$$

Then
$$\frac{|f(x+h)-f(x)|^p}{|h|^{\theta p+n}} \le 2^{p-1}\left(\frac{|a(x+h)-a(x)|^p}{|h|^{\theta p+n}} + \frac{|b(x+h)-b(x)|^p}{|h|^{\theta p+n}}\right.$$
so that
$$\int_{\mathbb{R}^n} \frac{|f(x+h)-f(x)|^p}{|h|^{\theta p+n}} dx$$
$$\le 2^{p-1}\int_{\mathbb{R}^n}\left(\frac{|a(x+h)-a(x)|^p}{|h|^{\theta p+n}} + \frac{|b(x+h)-b(x)|^p}{|h|^{\theta p+n}}\right)dx$$
$$\le 2^{2p-2}\frac{\|a\|_{L^p}^p}{|h|^{\theta p+n}} + 2^{p-1}\frac{|h|^p \| \, |Db| \, \|_{L^p}^p}{|h|^{\theta p+n}}$$
$$\le C_p|h|^{-\theta p-n}(\|a\|_{L^p} + |h| \, \|b\|_{W^{1,p}})^p \le C_p|h|^{-\theta p-n}K(|h|,f).$$
It follows that
$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|f(x+h)-f(x)|^p}{|h|^{\theta p+n}} dx\, dh$$
$$\le C_p\int_{\mathbb{R}^n} |h|^{-\theta p-n}K(|h|,f)^p dh$$
$$= C_p\int_0^\infty \frac{K(r,f)^p}{r^{\theta p+1}}\, dr \int_{\partial B(0,1)} d\sigma_{n-1} = C_{p,n}\|f\|_{\theta,p}^p.$$
Therefore, $(L^p(\mathbb{R}^n), W^{1,p}(\mathbb{R}^n))_{\theta,p}$ is continuously embedded in $W^{\theta,p}(\mathbb{R}^n)$. To be precise, we have estimated so far only the seminorm $[f]_{\theta,p}$. But we already know that each $(X,Y)_{\theta,p}$ is continuously embedded in $X+Y$; in our case $X+Y=X=L^p(\mathbb{R}^n)$ so that we have also $\|f\|_{L^p} \le C\|f\|_{\theta,p}$.

To prove the other embedding, for each $f \in W^{\theta,p}$ define a_t and b_t by (1.18). Then
$$\begin{aligned}\|a_t\|_{L^p}^p &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} |f(y)-f(x)|\frac{1}{t^n}\,\varphi\left(\frac{x-y}{t}\right)dy\right)^p dx \\ &\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(y)-f(x)|^p\frac{1}{t^n}\,\varphi\left(\frac{x-y}{t}\right)dydx.\end{aligned}$$
were for $p>1$ we applied the Hölder inequality to the product
$$|f(x)-f(y)|(t^{-n}\varphi((x-y)/t))^{1/p}\cdot(t^{-n}\varphi((x-y)/t))^{1-1/p}.$$
So we get
$$\int_0^\infty t^{-\theta p}\|a_t\|_{L^p}^p\frac{dt}{t} \le \int_0^\infty\left(\int_{\mathbb{R}^n\times\mathbb{R}^n} |f(y)-f(x)|^p\frac{1}{t^n}\varphi\left(\frac{x-y}{t}\right)dy\, dx\right)\frac{dt}{t}$$
$$= \int_{\mathbb{R}^n\times\mathbb{R}^n} |f(y)-f(x)|^p\left(\int_0^\infty t^{-\theta p}\frac{1}{t^n}\varphi\left(\frac{x-y}{t}\right)\frac{dt}{t}\right)dy\, dx$$
$$= \int_{\mathbb{R}^n\times\mathbb{R}^n} |f(y)-f(x)|^p\left(\int_{|x-y|}^\infty t^{-\theta p}\frac{1}{t^n}\varphi\left(\frac{x-y}{t}\right)\frac{dt}{t}\right)dy\, dx$$
$$\le \frac{\|\varphi\|_\infty}{\theta p+n}\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|f(y)-f(x)|^p}{|y-x|^{\theta p+n}}\, dx\, dy = C[f]_{W^{\theta,p}}^p.$$

Using (1.19) and arguing similarly, we get also

$$\int_0^\infty t^{(1-\theta)p} \|D_i b_t\|_{L^p}^p \frac{dt}{t} \leq \frac{C_i^{p-1}\|D_i\varphi\|_\infty}{\theta p + n} [f]^p_{W^{\theta,p}},$$

with $C_i = \int_{\mathbb{R}^n} |D_i\varphi(y)|dy$, while $\|b_t\|_{L^p} \leq \|f\|_{L^p}\|\varphi\|_{L^1} = \|f\|_{L^p}$.

Therefore, $t^{-\theta}K(t, f, L^p, W^{1,p}) \leq t^{-\theta}\|a_t\|_{L^p} + t^{1-\theta}\|b_t\|_{W^{1,p}} \in L^p_*(0,1)$, with norm estimated by $C\|f\|_{W^{\theta,p}}$, and the second part of the statement follows. □

Note that the proof of (1.16) yields also

$$(L^\infty(\mathbb{R}^n), Lip(\mathbb{R}^n))_{\theta,\infty} = (BUC(\mathbb{R}^n), BUC^1(\mathbb{R}^n))_{\theta,\infty} = C_b^\theta(\mathbb{R}^n).$$

We shall see later (§3.2) another method to prove (1.16) and (1.17).

Example 1.1.9 *Let $\Omega \in \mathbb{R}^n$ be an open set with the following property: there exists an extension operator E such that $E \in L(C_b(\overline{\Omega}), C_b(\mathbb{R}^n)) \cap L(C_b^1(\overline{\Omega}), C_b^1(\mathbb{R}^n))$, and also, for some $\theta \in (0,1)$, $E \in L(C_b^\theta(\overline{\Omega}), C_b^\theta(\mathbb{R}^n))$ (by extension operator we mean that $Ef_{|\overline{\Omega}} = f(x)$, for all $f \in C_b(\overline{\Omega})$). Then*

$$(C_b(\overline{\Omega}), C_b^1(\overline{\Omega}))_{\theta,\infty} = C_b^\theta(\overline{\Omega}).$$

Proof. Theorem 1.1.6 implies that

$$E \in L((C_b(\overline{\Omega}), C_b^1(\overline{\Omega}))_{\theta,\infty}, (C_b(\mathbb{R}^n), C_b^1(\mathbb{R}^n))_{\theta,\infty}).$$

We know already that $(C_b(\mathbb{R}^n), C_b^1(\mathbb{R}^n))_{\theta,\infty} = C_b^\theta(\mathbb{R}^n)$. So, for every $f \in (C_b(\overline{\Omega}), C_b^1(\overline{\Omega}))_{\theta,\infty}$ the extension Ef is in $C_b^\theta(\mathbb{R}^n)$ and $\|Ef\|_{C_b^\theta(\mathbb{R}^n)} \leq C\|f\|_{(C_b(\overline{\Omega}), C_b^1(\overline{\Omega}))_{\theta,\infty}}$. Since $f = Ef_{|\overline{\Omega}}$, then $f \in C_b^\theta(\overline{\Omega})$ and $\|f\|_{C_b^\theta(\overline{\Omega})} \leq C\|f\|_{(C_b(\overline{\Omega}), C_b^1(\overline{\Omega}))_{\theta,\infty}}$.

Conversely, if $f \in C_b^\theta(\overline{\Omega})$ then $Ef \in C_b^\theta(\mathbb{R}^n) = (C_b(\mathbb{R}^n), C_b^1(\mathbb{R}^n))_{\theta,\infty}$. The retraction operator $Rg = g_{|\overline{\Omega}}$ belongs obviously to $L(C_b(\mathbb{R}^n), C_b(\overline{\Omega})) \cap L(C_b^1(\mathbb{R}^n), C_b^1(\overline{\Omega}))$. Again by theorem 1.1.6, $f = R(Ef) \in (C_b(\overline{\Omega}), C_b^1(\overline{\Omega}))_{\theta,\infty}$, with norm not exceeding $C\|Ef\|_{C_b^\theta(\mathbb{R}^n)} \leq C'\|f\|_{C_b^\theta(\overline{\Omega})}$. □

Such a good extension operator exists if Ω is an open set with uniformly C^1 boundary. $\partial\Omega$ is said to be uniformly C^1 if there are $N \in \mathbb{N}$ and a (at most) countable set of balls B_k whose interior parts cover $\partial\Omega$, such that the intersection of more than N of these balls is empty, and diffeomorphisms $\varphi_k : B_k \mapsto B(0,1) \subset \mathbb{R}^n$ such that $\varphi_k(B_k \cap \overline{\Omega}) = \{y \in B(0,1) : y_n \geq 0\}$, and $\|\varphi_k\|_{C^1} + \|\varphi_k^{-1}\|_{C^1}$ are bounded by a constant independent of k. (In particular, each bounded Ω with C^1 boundary has uniformly C^1 boundary).

It is sufficient to construct E when $\Omega = \mathbb{R}^n_+$. The construction of E for any open set with uniformly C^1 boundary will follow by the usual method of local straightening the boundary.

If $\Omega = \mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^n : x_n > 0\}$ we may use the reflection method: we set

$$Ef(x) = \begin{cases} f(x), & x_n \geq 0, \\ \\ \alpha_1 f(x', -x_n) + \alpha_2 f(x', -2x_n), & x_n < 0, \end{cases}$$

where α_1, α_2 satisfy the continuity condition $\alpha_1 + \alpha_2 = 1$ and the differentiability condition $-\alpha_1 - 2\alpha_2 = 1$, that is $\alpha_1 = 3$, $\alpha_2 = -2$.

Then $E \in L(C(\overline{\mathbb{R}^n_+}), C_b(\mathbb{R}^n)) \cap L(C^\theta(\overline{\mathbb{R}^n_+}), C^\theta_b(\mathbb{R}^n)) \cap L(C^1(\overline{\mathbb{R}^n_+}), C^1_b(\mathbb{R}^n))$, for every $\theta \in (0,1)$.

Let now (Ω, μ) be a σ-finite measure space. To define the Lorentz spaces $L^{p,q}(\Omega)$ we introduce the rearrangements as follows. For every measurable $f : \Omega \mapsto \mathbb{R}$ or $f : \Omega \mapsto \mathbb{C}$ set

$$m(\sigma, f) = \mu\{x \in \Omega : |f(x)| > \sigma\}, \quad \sigma \geq 0,$$

and

$$f^*(t) = \inf\{\sigma : \, m(\sigma, f) \leq t\}, \quad t \geq 0.$$

Both $m(\cdot, f)$ and f^* are nonnegative, decreasing (i.e. nonincreasing), right continuous, and f^*, $|f|$ are equi-measurable, that is for each $\sigma_0 > 0$ we have

$$|\{t > 0 : f^*(t) > \sigma_0\}| = m(\sigma_0, f) = \mu\{x \in \Omega : |f(x)| > \sigma_0\},$$

and consequently $|\{t > 0 : f^*(t) \in [\sigma_1, \sigma_2]\}| = \mu\{x \in \Omega : |f(x)| \in [\sigma_1, \sigma_2]\}$, etc. Therefore, for each $p \geq 1$,

$$\int_\Omega |f(x)|^p \mu(dx) = \int_0^\infty (f^*(t))^p dt; \;\; \text{sup ess}\, |f(x)| = f^*(0) = \text{sup ess}\, f^*(t), \tag{1.20}$$

and for each measurable set $E \subset \Omega$,

$$\int_E |f(x)| \mu(dx) = \int_0^{\mu(E)} f^*(t) dt.$$

f^* is called the nonincreasing rearrangement of f onto $(0, \infty)$.

The Lorentz spaces $L^{p,q}(\Omega)$ $(1 \leq p \leq \infty, \; 1 \leq q \leq \infty)$ are defined by

$$L^{p,q}(\Omega) = \left\{ f \in L^1(\Omega) + L^\infty(\Omega) : \|f\|_{L^{p,q}} = \left(\int_0^\infty (t^{1/p} f^*(t))^q \frac{dt}{t} \right)^{1/q} < \infty \right\},$$

for $q < \infty$, and

$$L^{p,\infty}(\Omega) = \{ f \in L^1(\Omega) + L^\infty(\Omega) : \|f\|_{L^{p,\infty}} = \sup_{t>0} t^{1/p} f^*(t) < \infty \}.$$

(For $p = \infty$ we set as usual $1/\infty = 0$).

Note that in general $\| \cdot \|_{L^{p,q}}$ is not a norm but only a quasi-norm, i.e. the triangle inequality is replaced by $\|f + g\| \leq C(\|f\| + \|g\|)$. Moreover, due to (1.20),

$$L^{p,p}(\Omega) = L^p(\Omega), \quad 1 \leq p \leq \infty.$$

Example 1.1.10 *Let* (Ω, μ) *be a* σ*-finite measure space. Then* $(L^1(\Omega), L^\infty(\Omega))$ *is an interpolation couple. For* $0 < \theta < 1$, $1 \leq q \leq \infty$ *we have*

$$(L^1(\Omega), L^\infty(\Omega))_{\theta,q} = L^{\frac{1}{1-\theta}, q}(\Omega). \tag{1.21}$$

Proof. Let $\mathcal{V}$ be the space of all measurable, a.e. finitely valued (real or complex) functions defined in Ω. $\mathcal{V}$ is a linear topological Hausdorff space under convergence in measure on each measurable $E \subset \Omega$ with finite measure $\mu(E)$. Both $L^1(\Omega)$ and $L^\infty(\Omega)$ are continuously embedded in $\mathcal{V}$. Therefore $(L^1(\Omega), L^\infty(\Omega))$ is an interpolation couple.

The proof of (1.21) is based on the equality

$$K(t, f, L^1(\Omega), L^\infty(\Omega)) = \int_0^t f^*(s)ds, \quad t > 0. \tag{1.22}$$

Once (1.22) is established, (1.21) follows easily. Indeed, since f^* is decreasing then $K(t, f) \geq tf^*(t)$, so that for $q < \infty$

$$\begin{aligned}\|t^{-\theta}K(t,f)\|^q_{L^q_*} &= \int_0^\infty t^{-\theta q}K(t,f)^q \frac{dt}{t} \geq \int_0^\infty t^{-\theta q+q} f^*(t)^q \frac{dt}{t} \\ &= \|f\|^q_{L^{1/(1-\theta),q}(\Omega)},\end{aligned}$$

and similarly, for $q = \infty$

$$\sup_{t>0} \|t^{-\theta}K(t,f)\|_{L^\infty} \geq \sup_{t>0} \|t^{1-\theta}f^*(t)\|_{L^\infty} = \|f\|_{L^{1/(1-\theta),\infty}(\Omega)}.$$

The opposite inequality follows from the Hardy-Young inequality (A.10) (i) for $q < \infty$:

$$\|t^{-\theta}K(t,f)\|^q_{L^q_*} = \int_0^\infty t^{-\theta q}\left(\int_0^t sf^*(s)\frac{ds}{s}\right)^q \frac{dt}{t}$$

$$\leq \frac{1}{\theta^q}\int_0^\infty s^{(1-\theta)q}(f^*(s))^q \frac{ds}{s} = \|f\|^q_{L^{1/(1-\theta),q}(\Omega)},$$

and from the obvious inequality

$$\|t^{-\theta}K(t,f)\|_{L^\infty} \leq t^{-\theta}\int_0^t \frac{ds}{s^{1-\theta}}\|s^{1-\theta}f^*(s)\|_{L^\infty} = \frac{1}{\theta}\|f\|_{L^{1/(1-\theta),\infty}(\Omega)}$$

for $q = \infty$.

Let us prove (1.22). To prove $\leq$, for every $f \in L^1(\Omega) + L^\infty(\Omega)$ and $t > 0$, $x \in \Omega$ we set

$$a(x)\begin{cases} = f(x) - f^*(t)\dfrac{f(x)}{|f(x)|}, & \text{if } |f(x)| > f^*(t), \\ \\ = 0 & \text{otherwise,}\end{cases}$$

$$b(x) = f(x) - a(x).$$

The function a is defined in such a way that $|a(x)| = |f(x)| - f^*(t)$ if $|f(x)| > f^*(t)$, $|a(x)| = 0$ if $|f(x)| \leq f^*(t)$. Then

$$\|a\|_{L^1} = \int_E (|f(x)| - f^*(t))\mu(dx),$$

where $E = \{x \in \Omega : |f(x)| > f^*(t)\}$ has measure $\mu(E) = |\{s > 0 :> f^*(s) > f^*(t)\}|$ (because $|f|$ and f^* are equi-measurable) $= m(f^*(t), f) \leq t$, and f^* is constant in $[\mu(E), t]$. Therefore,

$$\|a\|_{L^1} = \int_0^{\mu(E)} (f^*(s) - f^*(t))ds \leq \int_0^t (f^*(s) - f^*(t))ds.$$

Moreover,

$$|b(x)| \begin{cases} = |f(x)| \text{ if } |f(x)| \leq f^*(t), \\ \\ = f^*(t) \text{ if } |f(x)| > f^*(t), \end{cases}$$

so that

$$|b(x)| \leq f^*(t) = \frac{1}{t}\int_0^t f^*(t)ds, \quad x \in \Omega.$$

Therefore,

$$K(t, f, L^1, L^\infty) \leq \|a\|_{L^1} + t\|b\|_{L^\infty} \leq \int_0^t f^*(s)ds.$$

To prove the opposite inequality we use the fact that for every decomposition $f = a+b$ we have (see exercise 7, §1.1.2)

$$f^*(s) \leq a^*((1-\varepsilon)s) + b^*(\varepsilon s), \quad s \geq 0, \ 0 < \varepsilon < 1.$$

Then, if $a \in L^1(\Omega)$, $b \in L^\infty(\Omega)$, and $a + b = f$ we have

$$\int_0^t f^*(s)ds \leq \int_0^t a^*((1-\varepsilon)s)ds + \int_0^t b^*(\varepsilon s)ds$$

$$\leq \frac{1}{1-\varepsilon}\int_0^\infty a^*(\tau)d\tau + tb^*(0) = \frac{1}{1-\varepsilon}\int_\Omega |a(x)|\mu(dx) + t \text{ sup ess } |b(x)|.$$

Letting $\varepsilon \to 0$ we get

$$\int_0^t f^*(s)ds \leq \|a\|_{L^1} + t\|b\|_{L^\infty},$$

so that

$$K(t, f, L^1(\Omega), L^\infty(\Omega)) \leq \int_0^t f^*(s)ds,$$

and the statement follows. □

1.1.2 Exercises

1) Prove that for every $x \in X + Y$, $t \mapsto K(t, x)$ is concave (and hence continuous) in $(0, \infty)$.

2) Prove that $x \mapsto \|x\|_{\theta,p}$ is a norm in $(X, Y)_{\theta,p}$, for each $\theta \in (0, 1)$, $p \in [1, \infty]$.

3) Take $\theta = 0$ in definition 1.1.1 and show that $(X, Y)_{0,p} = (X, Y)_0 = \{0\}$, for all $p \in [1, \infty)$. Show that $X \subset (X, Y)_{0,\infty}$.

Take $\theta = 1$ in definition 1.1.1 and show that $(X, Y)_{1,p} = (X, Y)_1 = \{0\}$, for all $p \in [1, \infty)$. Show that $Y \subset (X, Y)_{1,\infty}$.

4) Following the method of example 1.1.8 show that $(C_b(\mathbb{R}^n), C^1_b(\mathbb{R}^n))_{1,\infty} = Lip(\mathbb{R}^n)$, and that for $1 < p < \infty$, $(L^p(\mathbb{R}^n), W^{1,p}(\mathbb{R}^n))_{1,\infty} = W^{1,p}(\mathbb{R}^n)$.

5) Show that for $0 < \theta < 1$, $C^1_b(\mathbb{R}^n)$ is not dense in $C^\theta_b(\mathbb{R}^n)$. Show that $(C_b(\mathbb{R}^n), C^1_b(\mathbb{R}^n))_\theta$ is the space of the "little Hölder continuous" functions $h^\theta(\mathbb{R}^n)$, consisting of those bounded functions f such that

$$\lim_{h\to 0} \sup_{x\in\mathbb{R}^n} \frac{|f(x+h) - f(x)|}{|h|^\theta} = 0.$$

6) Following the method of example 1.1.8 show that

$$(L^p(\mathbb{R}^n), W^{1,p}(\mathbb{R}^n))_{\theta,q} = B^\theta_{p,q}(\mathbb{R}^n),$$

defined by $B^\theta_{p,q}(\mathbb{R}^n) = \{f \in L^p(\mathbb{R}^n) : [f]_{B^\theta_{p,q}} < \infty\}$, where

$$[f]_{B^\theta_{p,q}} = \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x) - f(x+h)|^p dx \right)^{q/p} \frac{1}{|h|^{\theta q + n}} dh \right)^{1/q},$$

and $\|f\|_{B^\theta_{p,q}} = \|f\|_{L^p} + [f]_{B^\theta_{p,q}}$.

7) Let Ω be an open set in $\mathbb{R}^n$ such that there exists an extension operator E belonging to $L(L^p(\Omega), L^p(\mathbb{R}^n)) \cap L(W^{\theta,p}(\Omega), W^{\theta,p}(\mathbb{R}^n)) \cap L(W^{1,p}(\Omega), W^{1,p}(\mathbb{R}^n))$, for some $p \in [1, \infty)$ and $\theta \in (0,1)$. Show that

$$(L^p(\Omega), W^{1,p}(\Omega))_{\theta,p} = W^{\theta,p}(\Omega).$$

Show that if Ω has uniformly C^1 boundary such extension operator E does exist (see the remarks after example 1.1.9).

The space $W^{\theta,p}(\Omega)$ is usually defined as the set of the functions $f \in L^p(\Omega)$ such that

$$[f]_{W^{\theta,p}} = \left(\int_{\Omega\times\Omega} \frac{|f(x) - f(y)|^p}{|x-y|^{\theta p + n}} \, dx\, dy \right)^{1/p} < \infty.$$

8) Let (Ω, μ) be any measure space. Prove that for each $a,\ b \in L^1(\Omega) + L^\infty(\Omega)$ we have

$$(a+b)^*(s) \le a^*((1+\varepsilon)s) + b^*(\varepsilon s), \quad s \ge 0,\ 0 < \varepsilon < 1.$$

(This is used in example 1.1.10). Hint: show preliminarly that

$$m(\sigma_0 + \sigma_1, a+b) \le m(\sigma_0, a) + m(\sigma_1, b), \quad \sigma_0,\ \sigma_1 \ge 0.$$

9) Prove that for each $x \in X + Y$ the function $K(\cdot, x)$ satisfies

$$K(t,x) \le \frac{t}{s} \, K(s,x) \ \ 0 < s < t.$$

1.2 The trace method

In this section we describe another construction of the real interpolation spaces, which will be useful for proving other properties, and will let us see the connection between interpolation theory and trace theory.

We shall use L^p and Sobolev spaces of functions with values in Banach spaces, whose definitions and elementary properties are in Appendix A.

Definition 1.2.1 *For $0 < \theta < 1$ and $1 \le p \le \infty$ define $V(p, \theta, Y, X)$ as the set of all functions $u : \mathbb{R}_+ \mapsto X + Y$ such that $u \in W^{1,p}(a, b; X + Y)$ for every $0 < a < b < \infty$, and*

$$t \mapsto u_\theta(t) = t^\theta u(t) \in L^p_*(0, +\infty; Y),$$

$$t \mapsto v_\theta(t) = t^\theta u'(t) \in L^p_*(0, +\infty; X),$$

with norm

$$\|u\|_{V(p,\theta,Y,X)} = \|u_\theta\|_{L^p_*(0,+\infty;Y)} + \|v_\theta\|_{L^p_*(0,+\infty;X)}.$$

Moreover, for $p = +\infty$ define a subspace of $V(\infty,\theta,Y,X)$, by

$$V_0(\infty,\theta,Y,X) = \{u \in V(\infty,\theta,Y,X) : \lim_{t\to 0}\|t^\theta u(t)\|_X = \lim_{t\to 0}\|t^\theta u'(t)\|_Y = 0\}.$$

It is not difficult to see that $V(p,\theta,Y,X)$ is a Banach space endowed with the norm $\|\cdot\|_{V(p,\theta,Y,X)}$, and that $V_0(\infty,\theta,Y,X)$ is a closed subspace of $V(\infty,\theta,Y,X)$. Moreover any function belonging to $V(p,\theta,Y,X)$ has a X-valued continuous extension at $t = 0$. Indeed, for $0 < s < t$ from the equality $u(t) - u(s) = \int_s^t u'(\sigma)d\sigma$ it follows, for $1 < p < \infty$,

$$\begin{aligned}\|u(t)-u(s)\|_X &\leq \left(\int_s^t \|\sigma^{\theta-1/p}u'(\sigma)\|_X^p d\sigma\right)^{1/p}\left(\int_s^t \sigma^{-(\theta-1/p)p'}d\sigma\right)^{1/p'}\\ &\leq \|u\|_{V(p,\theta,Y,X)}[p'(1-\theta)]^{-1/p'}(t^{p'(1-\theta)} - s^{p'(1-\theta)})^{1/p'},\end{aligned}$$

with $p' = p/(p-1)$. Arguing similarly, one sees that also if $p = 1$ or $p = \infty$, then u is uniformly continuous near $t = 0$.

With the aid of corollary A.3.1 we are able to characterize the real interpolation spaces as trace spaces.

Proposition 1.2.2 *For $(\theta,p) \in (0,1)\times[1,+\infty]$, $(X,Y)_{\theta,p}$ is the set of the traces at $t = 0$ of the functions in $V(p,1-\theta,Y,X)$, and the norm*

$$\|x\|^{Tr}_{\theta,p} = \inf\{\|u\|_{V(p,1-\theta,Y,X)} : \ x = u(0), \ u \in V(p,1-\theta,Y,X)\}$$

is an equivalent norm in $(X,Y)_{\theta,p}$. Moreover, for $0 < \theta < 1$, $(X,Y)_\theta$ is the set of the traces at $t = 0$ of the functions in $V_0(\infty,1-\theta,Y,X)$.

Proof. Let $x \in (X,Y)_{\theta,p}$. We need to define a function $u \in V(p,1-\theta,Y,X)$ such that $u(0) = x$.

For every $t > 0$ there are $a_t \in X$, $b_t \in Y$ such that $\|a_t\|_X + t\|b_t\|_Y \leq 2K(t,x)$. It holds $t^{1-\theta}\|b_t\|_Y \leq 2t^{-\theta}K(t,x)$, and the function $t \mapsto t^{-\theta}K(t,x)$ is in $L^p_*(0,+\infty)$. Moreover, we already know (see the proof of proposition 1.1.3) that $\lim_{t\to 0} b_t = x$ in $X+Y$. So, the function $t \mapsto b_t$ looks a good candidate for u. But in general it is not measurable with values in Y, and it is not in $W^{1,p}_{loc}(0,\infty)$ with values in X. So we have to modify it, and we proceed as follows.

For every $n \in \mathbb{N}$ let $a_n \in X$, $b_n \in Y$ be such that $a_n + b_n = x$, and

$$\|a_n\|_X + \frac{1}{n}\|b_n\|_Y \leq 2K(1/n,x).$$

For $t > 0$ set

$$u(t) = \sum_{n=1}^\infty b_{n+1}\chi_{\left(\frac{1}{n+1},\frac{1}{n}\right]}(t) = \sum_{n=1}^\infty (x - a_{n+1})\chi_{\left(\frac{1}{n+1},\frac{1}{n}\right]}(t),$$

where χ_I is the characteristic function of the interval I, and

$$v(t) = \frac{1}{t}\int_0^t u(s)ds.$$

Since $(X,Y)_{\theta,p}$ is contained in $(X,Y)_{\theta,\infty}$ then $t^{-\theta}K(t,x)$ is bounded, so that $\lim_{t\to 0} K(t,x) = 0$. Therefore, $\lim_{n\to\infty}\|a_n\|_X = 0$, so that $\|x-b_n\|_{X+Y} \le \|a_n\|_X \to 0$ as $n\to\infty$, and $x = \lim_{t\to 0} u(t) = \lim_{t\to 0} v(t)$ in $X+Y$. Moreover,

$$\|t^{1-\theta}u(t)\|_Y \le t^{1-\theta}\sum_{n=1}^{\infty}\chi_{\left(\frac{1}{n+1},\frac{1}{n}\right]}(t)2(n+1)K(1/(n+1),x) \le 4t^{-\theta}K(t,x), \tag{1.23}$$

so that $t\mapsto t^{1-\theta}u(t) \in L^p_*(0,+\infty;Y)$. By Corollary A.3.1, $t\mapsto t^{1-\theta}v(t)$ belongs to $L^p_*(0,+\infty;Y)$, and

$$\|t^{1-\theta}v\|_{L^p_*(0,+\infty;Y)} \le 4\theta^{-1}\|x\|_{\theta,p}.$$

On the other hand,

$$v(t) = x - \frac{1}{t}\int_0^t \sum_{n=1}^{\infty}\chi_{\left(\frac{1}{n+1},\frac{1}{n}\right]}(s)a_{n+1}ds,$$

so that v is differentiable almost everywhere with values in X, and

$$v'(t) = \frac{1}{t^2}\int_0^t g(s)ds - \frac{1}{t}g(t),$$

where $g(t) = \sum_{n=1}^{\infty}\chi_{\left(\frac{1}{n+1},\frac{1}{n}\right]}(t)a_{n+1}$ is such that

$$\|g(t)\|_X \le \sum_{n=1}^{\infty}\chi_{\left(\frac{1}{n+1},\frac{1}{n}\right]}(t)2K(1/(n+1),x) \le 2K(t,x).$$

It follows that

$$\|t^{1-\theta}v'(t)\| \le t^{-\theta}\sup_{0<s<t}\|g(s)\| + \|t^{-\theta}g(t)\| \le 4t^{-\theta}K(t,x). \tag{1.24}$$

Then $t\mapsto t^{1-\theta}v'(t)$ belongs to $L^p_*(0,+\infty;X)$, and

$$\|t^{1-\theta}v'\|_{L^p_*(0,+\infty;X)} \le 4\|x\|_{\theta,p}.$$

Therefore, x is the trace at $t=0$ of a function $v\in V(p,1-\theta,Y,X)$, and

$$\|x\|^{Tr}_{\theta,p} \le 2(2+1/\theta)\|x\|_{\theta,p}.$$

If $x\in(X,Y)_\theta$, then, by (1.23), $\lim_{t\to 0} t^{1-\theta}\|u(t)\|_Y = 0$, so that $\lim_{t\to 0} t^{1-\theta}\|v(t)\|_Y = 0$. By (1.24), $\lim_{t\to 0} t^{-\theta}\|g(t)\|_X = 0$, so $\lim_{t\to 0} t^{1-\theta}\|v'(t)\|_X = 0$. Then $v\in V_0(\infty,1-\theta,Y,X)$.

Conversely, let x be the trace at $t=0$ of a function $u\in V(p,1-\theta,Y,X)$. Then

$$x = x-u(t)+u(t) = -\int_0^t u'(s)ds + u(t) \;\; \forall t>0,$$

so that

$$t^{-\theta}K(t,x) \le t^{1-\theta}\left\|\frac{1}{t}\int_0^t u'(s)ds\right\|_X + t^{1-\theta}\|u(t)\|_Y. \tag{1.25}$$

Corollary A.3.1 implies now that $t\mapsto t^{-\theta}K(t,x)$ belongs to $L^p_*(0,+\infty)$, so that $x\in(X,Y)_{\theta,p}$, and

$$\|x\|_{\theta,p} \le \frac{1}{\theta}\,\|x\|^{Tr}_{\theta,p}.$$

If x is the trace of a function $u\in V_0(\infty,1-\theta,Y,X)$, we may assume without loss of generality that u vanishes for t large. Then, by (1.25), $\lim_{t\to 0} t^{-\theta}K(t,x) = \lim_{t\to\infty} t^{-\theta}K(t,x) = 0$, so that $x\in(X,Y)_\theta$. □

Example 1.2.3 Choosing $X = L^p(\mathbb{R}^n)$, $Y = W^{1,p}(\mathbb{R}^n)$, and $\theta = 1 - 1/p$, $1 < p < \infty$, we get the following well known characterization of $W^{1-1/p,p}(\mathbb{R}^n)$: $W^{1-1/p,p}(\mathbb{R}^n)$ is the space of the traces at $(x,0)$ of the functions $(x,t) \mapsto v(x,t) \in W^{1,p}(\mathbb{R}^{n+1}_+)$. Indeed, we already know that $W^{1-1/p,p}(\mathbb{R}^n) = (L^p(\mathbb{R}^n), W^{1,p}(\mathbb{R}^n))_{1-1/p,p}$, thanks to to example 1.1.8. By proposition 1.2.2, $W^{1-1/p,p}$ $(\mathbb{R}^n)$ is the space of the traces at $t = 0$ of the functions $v \in V(1/p, p, W^{1,p}(\mathbb{R}^n), L^p(\mathbb{R}^n))$. But $v \in V(1/p, p, W^{1,p}(\mathbb{R}^n),$ $L^p(\mathbb{R}^n))$ if and only if the function $v(x,t) = v(t)(x)$ is in $W^{1,p}(\mathbb{R}^{n+1}_+)$. Indeed, concerning measurability, it is possible to see that a function $w : (0,+\infty) \mapsto L^p(\mathbb{R}^n))$ (resp., $w : (0,+\infty) \mapsto W^{1,p}(\mathbb{R}^n))$ is measurable in the sense of definition A.1.1 iff $(t,x) \mapsto w(t,x)$ is measurable (resp., $(t,x) \mapsto w(t,x)$ and $(t,x) \mapsto D_i w(t,x)$ are measurable for all $i = 1, \dots n$). Concerning estimates, the condition $t \mapsto t^{1/p}v(t) \in L^p_*((0,+\infty), W^{1,p}(\mathbb{R}^n))$ means that

$$\int_0^{+\infty} \int_{\mathbb{R}^n} \left(|v(x,t)|^p + \sum_{i=1}^n |v_{x_i}(x,t)|^p \right) dx\, dt < \infty,$$

and the condition $t \mapsto t^{1/p}v'(t) \in L^p_*((0,+\infty), L^p(\mathbb{R}^n))$ means that v' is measurable with values in $L^p(\mathbb{R}^n))$ and

$$\int_0^{+\infty} \int_{\mathbb{R}^n} |v_t(x,t)|^p dx\, dt < \infty.$$

In particular, choosing $p = 2$ we get that $H^{1/2}(\mathbb{R}^n)$ is the space of the traces at $(x,0)$ of the functions $(x,t) \mapsto v(x,t) \in H^1(\mathbb{R}^{n+1}_+)$.

This example shows an important connection between interpolation theory and trace theory.

Remark 1.2.4 (important) 1. By Proposition 1.2.2, if $x \in (X,Y)_{\theta,p}$ or $x \in (X,Y)_\theta$, then x is the trace at $t = 0$ of a function u belonging to $L^p(a,b;Y) \cap W^{1,p}(a,b;X)$ for $0 < a < b$. But it is possible to find a more regular function $v \in V(p, 1-\theta, Y, X)$ (or $v \in V_0(\infty, 1-\theta, Y, X)$) such that $v(0) = x$. For instance we may take

$$v(t) = \frac{1}{t}\int_0^t u(s)ds, \quad t \geq 0.$$

Then $v \in W^{1,p}(a,b;Y) \cap W^{2,p}(a,b;X)$ for $0 < a < b$, $v(0) = x$, and moreover $t \mapsto t^{1-\theta}v(t)$ belongs to $L^p_*(0,+\infty;Y)$, $t \mapsto t^{2-\theta}v'(t)$ belongs to $L^p_*(0,+\infty;Y)$, and $t \mapsto t^{1-\theta}v'(t)$ belongs to $L^p_*(0,+\infty;X)$, with norms estimated by *const.* $\|u\|_{V(p,1-\theta,Y,X)}$.

Even better, choose any smooth nonnegative function $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}$, with compact support and $\int_0^\infty s^{-1}\varphi(s)ds = 1$, and set

$$v(t) = \int_0^\infty \varphi\left(\frac{t}{\tau}\right)u(\tau)\frac{d\tau}{\tau} = \int_0^\infty \varphi(s)u\left(\frac{t}{s}\right)\frac{ds}{s}.$$

Then $v \in C^\infty(\mathbb{R}_+; X \cap Y)$, $v(0) = x$, and

$$t \mapsto t^{n-\theta}v^{(n)}(t) \in L^p_*(0,+\infty;X), \quad n \in \mathbb{N},$$

$$t \mapsto t^{n+1-\theta}v^{(n)}(t) \in L^p_*(0,+\infty;Y), \quad n \in \mathbb{N} \cup \{0\},$$

with norms estimated by $c(n)\|u\|_{V(p,1-\theta,Y,X)}$. If in addition $p = \infty$ and $x \in (X,Y)_\theta$ then

$$\lim_{t\to 0} t^{n-\theta}\|v^{(n)}(t)\|_X = 0, \quad n \in \mathbb{N},$$

$$\lim_{t\to 0} t^{n+1-\theta}\|v^{(n)}(t)\|_Y = 0. \quad n \in \mathbb{N} \cup \{0\},$$

2. Let $x \in X+Y$ be the trace at $t=0$ of a function $v \in V(p, 1-\theta, Y, X)$. Fix any $\varphi \in C_0^\infty([0,+\infty))$ such that $\varphi \equiv 1$ in a right neighborhood of 0, say in $(0,1]$. The function $t \mapsto \varphi(t)v(t)$ is in $V(p,1-\theta,Y,X)$, its norm does not exceed $C\|v\|_{V(p,1-\theta,Y,X)}$, with C depending only on φ, and its trace at $t=0$ is still x. Moreover, it has compact support in $[0,+\infty)$. This shows that in the defintion of the trace spaces one could consider just the subset of $V(p,1-\theta,Y,X)$ consisting of the functions with compact support, obtaining an equivalent trace space (i.e., the same space with an equivalent norm).

By means of the trace method it is easy to prove some important density properties.

Proposition 1.2.5 *Let $0<\theta<1$. For $1 \le p < \infty$, $X \cap Y$ is dense in $(X,Y)_{\theta,p}$. For $p=\infty$, $(X,Y)_\theta$ is the closure of $X\cap Y$ in $(X,Y)_{\theta,\infty}$.*

Proof. Let $p<\infty$, and let $x \in (X,Y)_{\theta,p}$. By Remark 1.2.4, $x = v(0)$, where $v \in C^\infty((0,\infty); X\cap Y) \cap V(p,1-\theta,Y,X)$, and moreover $t \mapsto t^{2-\theta}v' \in L^p_*(0,+\infty;Y)$. Set

$$x_\varepsilon = v(\varepsilon), \quad \forall \varepsilon > 0.$$

Then $x_\varepsilon \in X\cap Y$, and we shall show that $x_\varepsilon \to x$ in $(X,Y)_{\theta,p}$.

We have $x_\varepsilon - x = z_\varepsilon(0)$, where

$$z_\varepsilon(t) = (v(\varepsilon)-v(t))\chi_{[0,\varepsilon]}(t).$$

It is not hard to check that $z_\varepsilon \in W^{1,p}(a,b;X)$ for $0<a<b<\infty$, and that $z_\varepsilon'(t) = -v'(t)\chi_{(0,\varepsilon)}(t)$. It follows that

$$\lim_{\varepsilon\to 0} \|t^{1-\theta}z_\varepsilon'(t)\|_{L^p_*(0,+\infty;X)} = 0.$$

Moreover, due to the equality

$$z_\varepsilon(t) = \int_t^{+\infty} \chi_{(0,\varepsilon)}(s)v'(s)ds,$$

we get, using the Hardy-Young inequality (A.10)(ii),

$$\|t^{1-\theta}z_\varepsilon(t)\|_{L^p_*(0,+\infty;Y)} \le \left(\int_0^{+\infty} t^{(1-\theta)p}\left(\int_t^{+\infty}\chi_{(0,\varepsilon)}(s)s\|v'(s)\|_Y\frac{ds}{s}\right)^p\frac{dt}{t}\right)^{1/p}$$

$$\le \frac{1}{1-\theta}\left(\int_0^{+\infty}\chi_{(0,\varepsilon)}(s)s^{(2-\theta)p}\|v'(s)\|_Y\frac{ds}{s}\right)^{1/p},$$

so that $t\mapsto t^{1-\theta}z_\varepsilon(t) \in L^p_*(0,+\infty;Y)$ for every ε, and

$$\lim_{\varepsilon\to 0}\|t^{1-\theta}z_\varepsilon(t)\|_{L^p_*(0,+\infty;Y)} = 0.$$

Therefore, $z_\varepsilon \to 0$ in $V(p,1-\theta,Y,X)$ as $\varepsilon\to 0$, which means that $\|x_\varepsilon - x\|^T_{\theta,p} \to 0$ as $\varepsilon \to 0$. From Proposition 1.2.2 we get $\lim_{\varepsilon\to 0}\|x_\varepsilon - x\|_{\theta,p} = 0$.

Let now $x \in (X,Y)_\theta$. Due again to remark 1.2.4, x is the trace at $t=0$ of a function $v \in V_0(\infty,1-\theta,Y,X)$, such that $t\mapsto t^{2-\theta}v'(t) \in L^\infty(0,+\infty;Y)$ and $\lim_{t\to 0}t^{2-\theta}\|v'(t)\|_Y = 0$. Let x_ε, z_ε be defined as above. Then $\lim_{t\to 0}t^{1-\theta}\|v'(t)\|_X = 0$, so that

$$\sup_{t>0} t^{1-\theta}\|z_\varepsilon'(t)\|_X = \sup_{0<t\le\varepsilon} t^{1-\theta}\|v'(t)\|_X \to 0, \quad \text{as } \varepsilon\to 0,$$

and
$$\sup_{t>0} t^{1-\theta}\|z_\varepsilon(t)\|_Y = \sup_{0<t\le\varepsilon} t^{1-\theta}\|v(\varepsilon)-v(t)\|_Y \le 2\sup_{0<s\le\varepsilon} s^{1-\theta}\|v(s)\|_Y \to 0,$$
as $\varepsilon \to 0$. Arguing as before, it follows that $\|x_\varepsilon - x\|_{\theta,\infty} \to 0$ as $\varepsilon \to 0$. □

1.2.1 Exercises

1) Prove the statements of remark 1.2.4.

2) Prove that $(X,Y)_{\theta,p}$ is the set of the elements $x \in X+Y$ such that $x = u(t)+v(t)$ for almost all $t>0$, with $t \mapsto t^{-\theta}u(t) \in L^p_*(0,\infty;X)$, $t \mapsto t^{1-\theta}u(t) \in L^p_*(0,\infty;Y)$, and the norm
$$x \mapsto \inf_{x=u(t)+v(t)} \|t^\theta u(t)\|_{L^p_*(0,\infty;X)} + \|t^{1-\theta}v(t)\|_{L^p_*(0,\infty;Y)}$$
is equivalent to the norm of $(X,Y)_{\theta,p}$.

3) Prove that $(X,Y)_{\theta,p}$ is the set of the elements $x \in X+Y$ such that $x = \int_0^\infty u(t)dt/t$, with $t \mapsto t^{-\theta}u(t) \in L^p_*(0,\infty;X)$, $t \mapsto t^{1-\theta}u(t) \in L^p_*(0,\infty;Y)$, and the norm
$$x \mapsto \inf_{x=\int_0^\infty u(t)dt/t} \|t^\theta u(t)\|_{L^p_*(0,\infty;X)} + \|t^{1-\theta}v(t)\|_{L^p_*(0,\infty;Y)}$$
is equivalent to the norm of $(X,Y)_{\theta,p}$.

Hint: use remark 1.2.4 to write any $x \in (X,Y)_{\theta,p}$ as $x = -\int_0^\infty v(t)dt$, as in the proof of next proposition 1.3.2.

4) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^1 boundary. Prove that for $1<p<\infty$, the space $W^{1-1/p,p}(\partial\Omega)$ is the space of the traces on $\partial\Omega$ of the functions in $W^{1,p}(\Omega)$.

1.3 Intermediate spaces and reiteration

Let us introduce two classes of intermediate spaces for the interpolation couple (X,Y).

Definition 1.3.1 *Let $0 \le \theta \le 1$, and let E be a Banach space such that $X \cap Y \subset E \subset X+Y$.*

(i) E is said to belong to the class J_θ between X and Y if there is a constant c such that
$$\|x\|_E \le c\|x\|_X^{1-\theta}\|x\|_Y^\theta, \quad \forall x \in X \cap Y.$$
In this case we write $E \in J_\theta(X,Y)$.

(ii) E is said to belong to the class K_θ between X and Y if there is $k>0$ such that
$$K(t,x) \le kt^\theta\|x\|_E, \quad \forall x \in E,\ t>0.$$
In this case we write $E \in K_\theta(X,Y)$.

If $\theta \in (0,1)$ this means that E is continuously embedded in $(X,Y)_{\theta,\infty}$.

A useful characterization of $J_\theta(X,Y)$ is the following one.

Proposition 1.3.2 *Let $0 < \theta < 1$, and let E be a Banach space such that $X \cap Y \subset E \subset X + Y$. The following statements are equivalent:*

(i) *E belongs to the class J_θ between X and Y,*

(ii) *$(X,Y)_{\theta,1} \subset E$.*

Proof. The implication (ii) $\Rightarrow$ (i) is a straightforward consequence of Corollary 1.1.7, with $p = 1$. Let us show that (i) $\Rightarrow$ (ii). For every $x \in (X,Y)_{\theta,1}$, let $u \in V(1, 1-\theta, Y, X)$ be such that $u(t)$ vanishes for large t, $u(0) = x$, and set

$$v(t) = \frac{1}{t}\int_0^t u(s)ds.$$

By remark 1.2.4, $t \mapsto t^{2-\theta}v'(t)$ belongs to $L^1_*(0,+\infty;Y)$, and $t \mapsto t^{1-\theta}v'(t)$ belongs to $L^1_*(0,+\infty;X)$. Moreover $v(0) = x$, $v(+\infty) = 0$, so that

$$x = -\int_0^{+\infty} v'(t)dt.$$

Let c be such that $\|y\|_E \leq c\|y\|_Y^\theta \|y\|_X^{1-\theta}$ for every $y \in X \cap Y$. Then

$$\|v'(t)\|_E \leq c\|v'(t)\|_Y^\theta \|v'(t)\|_X^{1-\theta} = ct^{-1}\|t^{2-\theta}v'(t)\|_Y^\theta \|t^{1-\theta}v'(t)\|_X^{1-\theta}.$$

Since the function $t \mapsto \|t^{2-\theta}v'(t)\|_Y^\theta$ belongs to $L^{1/\theta}_*(0,+\infty)$ and $t \mapsto \|t^{1-\theta}v'(t)\|_X^{1-\theta}$ belongs to $L^{1/(1-\theta)}_*(0,+\infty)$, by the Hölder inequality $v' \in L^1(0,+\infty;E)$, and

$$\|x\|_E \leq c(\|t^{2-\theta}v'(t)\|_{L^1_*(0,\infty;Y)})^\theta (\|t^{1-\theta}v'(t)\|_{L^1_*(0,\infty;X)})^{1-\theta} \leq const.\, \|x\|_{\theta,1}.$$

□

By Definition 1.3.1 and Proposition 1.2.5, if $0 < \theta < 1$ a space E belongs to $K_\theta(X,Y)$ $\cap$ $J_\theta(X,Y)$ if and only if

$$(X,Y)_{\theta,1} \subset E \subset (X,Y)_{\theta,\infty}.$$

In particular, $(X,Y)_{\theta,p}$ and $(X,Y)_\theta$ are in $K_\theta(X,Y) \cap J_\theta(X,Y)$, for every $p \in [1,\infty]$. We shall see in chapter 2 that also the complex interpolation spaces $[X,Y]_\theta$ are in $K_\theta(X,Y)$ $\cap$ $J_\theta(X,Y)$.

But there are also intermediate spaces belonging to $K_\theta(X,Y) \cap J_\theta(X,Y)$ which are not interpolation spaces.

Example 1.3.3 $C^1_b(\mathbb{R}^n) \in J_{1/2}(C_b(\mathbb{R}^n), C^2_b(\mathbb{R}^n)) \cap K_{1/2}(C_b(\mathbb{R}^n), C^2_b(\mathbb{R}^n))$. *But* $C^1_b(\mathbb{R}^n)$ *is not an interpolation space between* $C_b(\mathbb{R}^n)$ *and* $C^2_b(\mathbb{R}^n)$.

Proof. From the inequalities ($i = 1, \ldots, n$)

$$|f(x+he_i) - f(x) - D_if(x)h| \leq \frac{1}{2}\|D_{ii}f\|_\infty h^2, \;\; \forall x \in \mathbb{R}^n, \; h > 0,$$

we get

$$|D_if(x)| \leq \frac{|f(x+he_i) - f(x)|}{h} + \frac{1}{2}\|D_{ii}f\|_\infty h, \;\; \forall x \in \mathbb{R}^n, \; h > 0,$$

so that

$$\|D_i f\|_\infty \le \frac{2\|f\|_\infty}{h} + \frac{1}{2}\|D_{ii} f\|_\infty h, \quad \forall h > 0.$$

Taking the minimum on h over $(0, +\infty)$ we get

$$\|D_i f\|_\infty \le 2(\|f\|_\infty)^{1/2}(\|D_{ii} f\|_\infty)^{1/2}, \quad \forall f \in C_b^2(\mathbb{R}^n)$$

so that

$$\|f\|_{C_b^1} \le (\|f\|_\infty)^{1/2}\left((\|f\|_\infty)^{1/2} + 2\textstyle\sum_{i=1}^n (\|D_{ii} f\|_\infty)^{1/2}\right)$$

$$\le C(\|f\|_\infty)^{1/2}(\|f\|_{C_b^2})^{1/2}.$$

This implies that $C_b^1(\mathbb{R}^n)$ belongs to $J_{1/2}(C_b(\mathbb{R}^n), C_b^2(\mathbb{R}^n))$. To prove that it belongs also to $K_{1/2}(C_b(\mathbb{R}^n), C_b^2(\mathbb{R}^n))$, namely that it is continuously embedded in $(C_b(\mathbb{R}^n), C_b^2(\mathbb{R}^n))_{1/2,\infty}$, we argue as in example 1.1.8: for every $f \in C_b^1(\mathbb{R}^n)$ the functions a_t, b_t defined in (1.18) are easily seen to satisfy

$$\|a_t\|_\infty \le Ct[f]_{Lip}, \quad \|b_t\|_{C_b^1} \le C\|f\|_{C_b^1}, \quad \|D_{ij} b_t\|_\infty \le Ct^{-1}[f]_{Lip}.$$

Therefore, $K(t, f, C_b(\mathbb{R}^n), C_b^2(\mathbb{R}^n)) \le \|a_{t^{1/2}}\|_\infty + t\|b_{t^{1/2}}\|_{C_b^2} \le Ct^{1/2}\|f\|_{C_b^1}$ so that $C_b^1(\mathbb{R}^n)$ is in $K_{1/2}(C_b(\mathbb{R}^n), C_b^2(\mathbb{R}^n))$.

But $C_b^1(\mathbb{R}^n)$ is not a real interpolation space between $C_b(\mathbb{R}^n)$ and $C_b^2(\mathbb{R}^n)$, even for $n = 1$. More precisely, there does not exist any constant C such that $\|T\|_{L(C_b^1(\mathbb{R}))} \le C(\|T\|_{L(C_b^2(\mathbb{R}))})^{1/2}(\|T\|_{L(C_b(\mathbb{R}))})^{1/2}$ for all $T \in L(C_b^2(\mathbb{R})) \cap L(C_b(\mathbb{R}))$.

Indeed, consider the family of operators

$$(T_\varepsilon f)(x) = \int_{-1}^1 \frac{x}{x^2 + y^2 + \varepsilon^2}(f(y) - f(0))\, dy, \quad x \in \mathbb{R}.$$

It is easy to see that $\|T_\varepsilon\|_{L(C_b(\mathbb{R}))}$ and $\|T_\varepsilon\|_{L(C_b^2(\mathbb{R}))}$ are bounded by a constant independent of ε. Indeed, for every continuous and bounded f,

$$|(T_\varepsilon f)(x)| \le 2\int_{-1}^1 \frac{|x|}{x^2 + y^2 + \varepsilon^2}\|f\|_\infty dy \le 2\pi\|f\|_\infty,$$

$$(T_\varepsilon f)'(x) = \int_{-1}^1 \frac{-x^2 + y^2 + \varepsilon^2}{(x^2 + y^2 + \varepsilon^2)^2}\,(f(y) - f(0))dy,$$

and for every $f \in C_b^1(\mathbb{R})$,

$$(T_\varepsilon f)''(x) = \int_{-1}^1 \frac{-2x(-x^2 + 3y^2 + 3\varepsilon^2)}{(x^2 + y^2 + \varepsilon^2)^3}\int_0^y f'(s)ds\, dy$$

$$= \int_{-1}^1 \frac{-2x(-x^2 + 3y^2 + 3\varepsilon^2)}{(x^2 + y^2 + \varepsilon^2)^3}\int_0^y (f'(s) - f'(0))ds\, dy,$$

so that, if $f \in C_b^2(\mathbb{R})$,

$$|(T_\varepsilon f)''(x)| \le |x|\int_{-1}^1 \frac{x^2 + 3y^2 + 3\varepsilon^2}{(x^2 + y^2 + \varepsilon^2)^2}\; dy\,\|f''\|_\infty \le 3\pi\|f''\|_\infty.$$

On the contrary, choosing $f_\varepsilon(x) = (x^2+\varepsilon^2)^{1/2}\eta(x)$, with $\eta \in C_0^\infty(\mathbb{R})$, $\eta \equiv 1$ in $[-1,1]$, we get

$$(T_\varepsilon f_\varepsilon)'(0) = \int_{-1}^{1} \frac{(y^2+\varepsilon^2)^{1/2}-\varepsilon}{y^2+\varepsilon^2}\, dy = \frac{1}{\varepsilon}\int_{-1/\varepsilon}^{1/\varepsilon} \frac{(s^2+1)^{1/2}-1}{s^2+1}\, ds$$

so that $\lim_{\varepsilon\to 0}(T_\varepsilon f_\varepsilon)'(0) = +\infty$, while the C^1_b norm of f_ε is bounded by a constant independent of ε. Therefore $\|T_\varepsilon\|_{L(C^1_b(\mathbb{R}))}$ blows up as $\varepsilon \to 0$. By theorem 1.1.6, $C^1_b(\mathbb{R})$ cannot be a real interpolation space between $C_b(\mathbb{R})$ and $C^2_b(\mathbb{R})$.

This counterexample is due to Mitjagin and Semenov, it shows also that $C^1([-1,1])$ is not a real interpolation space between $C([-1,1])$ and $C^2([-1,1])$, and it may be obviously adapted to show that for any dimension n, $C^1_b(\mathbb{R}^n)$ is not a real interpolation space between $C_b(\mathbb{R}^n)$ and $C^2_b(\mathbb{R}^n)$. □

Remark 1.3.4 Arguing similarly one sees that $C^k_b(\mathbb{R}^n)$ is in $J_{1/2}(C^{k-1}_b(\mathbb{R}^n), C^{k+1}_b(\mathbb{R}^n))$ $\cap\, K_{1/2}(C^{k-1}_b(\mathbb{R}^n), C^{k+1}_b(\mathbb{R}^n))$, for every $k \in \mathbb{N}$. It follows easily that for $m_1 < k < m_2 \in \mathbb{N}$, $C^k_b(\mathbb{R}^n)$ belongs to $J_{(k-m_1)/(m_2-m_1)}(C^{m_1}_b(\mathbb{R}^n),\, C^{m_2}_b(\mathbb{R}^n))$. For instance, knowing that $C^1_b(\mathbb{R}^n)$ belongs to $J_{1/2}(C_b(\mathbb{R}^n), C^2_b(\mathbb{R}^n))$ and $C^2_b(\mathbb{R}^n)$ belongs to $J_{1/2}(C^1_b(\mathbb{R}^n), C^3_b(\mathbb{R}^n))$ one gets, for every $f \in C^3_b(\mathbb{R}^n)$,

$$\|f\|_{C^1_b} \le C\|f\|_\infty^{1/2}\|f\|_{C^2_b}^{1/2} \le C'\|f\|_\infty^{1/2}(\|f\|_{C^1_b}^{1/2}\|f\|_{C^3_b}^{1/2})^{1/2}$$

so that $\|f\|_{C^1_b}^{3/4} \le C'\|f\|_\infty^{1/2}\|f\|_{C^3_b}^{1/4}$, which implies

$$\|f\|_{C^1_b} \le C''\|f\|_\infty^{2/3}\|f\|_{C^3_b}^{1/3}$$

that is, $C^1_b(\mathbb{R}^n)$ belongs to $J_{1/3}(C_b(\mathbb{R}^n), C^3_b(\mathbb{R}^n))$.

Now we are able to state the Reiteration Theorem. It is one of the main tools of general interpolation theory.

Theorem 1.3.5 *Let $0 \le \theta_0, \theta_1 \le 1$, $\theta_0 \ne \theta_1$. Fix $\theta \in (0,1)$ and set $\omega = (1-\theta)\theta_0 + \theta\theta_1$. The following statements hold true.*

(i) If E_i belong to the class K_{θ_i} $(i = 0,1)$ between X and Y, then

$$(E_0, E_1)_{\theta,p} \subset (X,Y)_{\omega,p}, \quad \forall p \in [1,\infty], \quad (E_0,E_1)_\theta \subset (X,Y)_\omega.$$

(ii) If E_i belong to the class J_{θ_i} $(i = 0,1)$ between X and Y, then

$$(X,Y)_{\omega,p} \subset (E_0,E_1)_{\theta,p}, \quad \forall p \in [1,\infty], \quad (X,Y)_\omega \subset (E_0,E_1)_\theta.$$

Consequently, if E_i belong to $K_{\theta_i}(X,Y) \cap J_{\theta_i}(X,Y)$, then

$$(E_0,E_1)_{\theta,p} = (X,Y)_{\omega,p}, \quad \forall p \in [1,\infty], \quad (E_0,E_1)_\theta = (X,Y)_\omega,$$

with equivalence of the respective norms.

Proof. Without loss of generality (recalling that $(E_1, E_0)_{\theta,p} = (E_0, E_1)_{1-\theta,p}$ and $(E_1, E_0)_\theta = (E_0, E_1)_{1-\theta}$) we may assume that $\theta_0 < \theta_1$.

Let us prove statement (i). Let k_i be such that $K(t, x) \le k_i t^{\theta_i} \|x\|_{E_i}$ for every $x \in E_i$, $i = 0, 1$. For each $x \in (E_0, E_1)_{\theta,p}$, let $a \in E_0$, $b \in E_1$ be such that $x = a + b$. Then

$$K(t, x, X, Y) \le K(t, a, X, Y) + K(t, b, X, Y) \le k_0 t^{\theta_0} \|a\|_{E_0} + k_1 t^{\theta_1} \|b\|_{E_1}.$$

Since a and b are arbitrary, it follows that

$$K(t, x, X, Y) \le \max\{k_0, k_1\} t^{\theta_0} K(t^{\theta_1-\theta_0}, x, E_0, E_1).$$

Consequently,

$$t^{-\omega} K(t, x, X, Y) \le \max\{k_0, k_1\} t^{-\theta(\theta_1-\theta_0)} K(t^{\theta_1-\theta_0}, x, E_0, E_1). \tag{1.26}$$

By the change of variable $s = t^{\theta_1-\theta_0}$ we see that $t \mapsto t^{-\omega} K(t, x, X, Y)$ is in $L^p_*(0, +\infty)$, which means that x belongs to $(X, Y)_{\omega,p}$, and

$$\begin{cases} \|x\|_{(X,Y)_{\omega,p}} \le \max\{k_0, k_1\}(\theta_1 - \theta_0)^{-1/p} \|x\|_{(E_0,E_1)_{\theta,p}}, & \textit{if } p < \infty, \\ \\ \|x\|_{(X,Y)_{\omega,\infty}} \le \max\{k_0, k_1\} \|x\|_{(E_0,E_1)_{\theta,p}}, & \textit{if } p = \infty. \end{cases}$$

If $x \in (E_0, E_1)_\theta$, by (1.26) we get

$$\lim_{t\to 0} t^{-\omega} K(t, x, X, Y) \le \max\{k_0, k_1\} \lim_{s\to 0} s^{-\theta} K(s, x, E_0, E_1) = 0,$$

and

$$\lim_{t\to +\infty} t^{-\omega} K(t, x, X, Y) \le \max\{k_0, k_1\} \lim_{s\to +\infty} s^{-\theta} K(s, x, E_0, E_1) = 0,$$

so that $x \in (X, Y)_\omega$.

Let us prove statement (ii). By proposition 1.2.2 and remark 1.2.4, every $x \in (X, Y)_{\omega,p}$ is the trace at $t = 0$ of a regular function $v : \mathbb{R}_+ \mapsto X \cap Y$ such that $v(+\infty) = 0$, $t \mapsto t^{1-\omega} v'(t)$ belongs to $L^p_*(0, +\infty, X)$, $t \mapsto t^{2-\omega} v'(t)$ belongs to $L^p_*(0, +\infty, Y)$, and

$$\|t^{1-\omega} v'(t)\|_{L^p_*(0,+\infty,X)} + \|t^{2-\omega} v'(t)\|_{L^p_*(0,+\infty,Y)} \le k \|x\|^{Tr}_{(X,Y)_{\omega,p}},$$

with k independent of x and v. We shall show that the function

$$g(t) = v(t^{1/(\theta_1-\theta_0)}), \quad t > 0,$$

belongs to $V(p, 1 - \theta, E_0, E_1)$: since $g(0) = x$, this will imply, through proposition 1.2.2, that $x \in (E_0, E_1)_{\theta,p}$.

To this aim we preliminarly estimate $\|v'(t)\|_{E_i}$, $i = 0, 1$. Let c_i be such that

$$\|y\|_{E_i} \le c_i \|y\|_X^{1-\theta_i} \|y\|_Y^{\theta_i} \;\; \forall y \in Y, \quad i = 0, 1.$$

Then

$$\|v'(s)\|_{E_i} \le \frac{c_i}{s^{\theta_i+1-\omega}} \|s^{1-\omega} v'(s)\|_X^{1-\theta_i} \|s^{2-\omega} v'(s)\|_Y^{\theta_i}, \quad i = 0,\ 1,$$

so that from the equalities

$$\theta_0 + 1 - \omega = 1 - \theta(\theta_1 - \theta_0), \quad \theta_1 + 1 - \omega = 1 + (1 - \theta)(\theta_1 - \theta_0),$$

we get

$$\begin{cases} (i) \quad \|s^{1-\theta(\theta_1-\theta_0)}v'(s)\|_{L^p_*(0,+\infty;E_0)} \leq c_0 k\, \|x\|^{Tr}_{(X,Y)_{\omega,p}}, \\ \\ (ii) \quad \|s^{1+(1-\theta)(\theta_1-\theta_0)}v'(s)\|_{L^p_*(0,+\infty;E_1)} \leq c_0 k\, \|x\|^{Tr}_{(X,Y)_{\omega,p}}. \end{cases} \tag{1.27}$$

From the equality $v(t) = -\int_t^\infty v'(s)ds$ and 1.27(ii), using the Hardy-Young inequality (A.10)(ii) if $p < \infty$, we get

$$\|t^{(1-\theta)(\theta_1-\theta_0)}v(t)\|_{L^p_*(0,+\infty;E_1)} \leq \frac{c_0 k}{(1-\theta)(\theta_1-\theta_0)}\|x\|^{Tr}_{(X,Y)_{\omega,p}}.$$

It follows that $t \mapsto t^{1-\theta}g(t) \in L^p_*(0,+\infty;E_1)$, and

$$\|t^{1-\theta}g(t)\|_{L^p_*(0,+\infty;E_1)} \leq (\theta_1-\theta_0)^{-1/p}\|t^{(1-\theta)(\theta_1-\theta_0)}v(t)\|_{L^p_*(0,+\infty;E_1)}.$$

Moreover $g'(t) = (\theta_1-\theta_0)^{-1}t^{-1+1/(\theta_1-\theta_0)}v'(t^{1/(\theta_1-\theta_0)})$, so that, by (1.27)(i), $t \mapsto t^{1-\theta}g'(t) = (\theta_1-\theta_0)^{-1}t^{(1-\theta(\theta_1-\theta_0))/(\theta_1-\theta_0)}v'(t^{1/(\theta_1-\theta_0)}) \in L^p_*(0,+\infty;E_0)$, and

$$\|t^{1-\theta}g'(t)\|_{L^p_*(0,+\infty;E_0)} \leq (\theta_1-\theta_0)^{-1-1/p}\|t^{1-\theta(\theta_1-\theta_0)}v'(t)\|_{L^p_*(0,+\infty;E_0)}.$$

Therefore, $g \in V(p, 1-\theta, E_0, E_1)$, so that $x = g(0)$ belongs to $(E_0, E_1)_{\theta,p}$, and

$$\|x\|^T_{(E_0,E_1)_{\theta,p}} \leq (\theta_1-\theta_0)^{-1-1/p}k\|x\|^T_{(X,Y)_{\omega,p}}.$$

If $x \in (X,Y)_\omega$, then (1.27)(i) has to be replaced by

$$\lim_{s\to 0} s^{1-\theta(\theta_1-\theta_0)}\|v'(s)\|_{E_0} = 0,$$

so that

$$\lim_{t\to 0} t^{1-\theta}\|g'(t)\|_{E_0} = \lim_{t\to 0}\frac{t^{-\theta+1/(\theta_1-\theta_0)}}{\theta_1-\theta_0}\|v'(t^{1/(\theta_1-\theta_0)})\|_{E_0} = 0.$$

Similarly, (1.27)(ii) has to be replaced by

$$\lim_{s\to 0} s^{1+(1-\theta)(\theta_1-\theta_0)}\|v'(s)\|_{E_1} = 0.$$

Using the equality

$$t^{1-\theta}g(t) = t^{1-\theta}\int_{t^{1/(\theta_1-\theta_0)}}^{\varepsilon^{1/(\theta_1-\theta_0)}} v'(s)ds + \frac{t^{1-\theta}}{\varepsilon^{1-\theta}}\left(\varepsilon^{1-\theta}\int_{\varepsilon^{1/(\theta_1-\theta_0)}}^{+\infty} v'(s)ds\right),$$

which holds for $0 < t < \varepsilon$, one deduces that $\lim_{t\to 0} t^{1-\theta}\|g(t)\|_{E_1} = 0$. □

Remark 1.3.6 The assumption $\theta_0 \neq \theta_1$ is not removable. Consider for instance the case $E_0 = E_1 = (X,Y)_{\theta_0,\infty}$, which is in $K_{\theta_0}(X,Y)$. If statement (i) of the theorem would be true for $\theta_0 = \theta_1$ then $(X,Y)_{\theta_0,\infty} \subset (X,Y)_{\theta_0,p}$ for each $p < \infty$, which is false in general.

Remark 1.3.7 By proposition 1.2.3, $(X,Y)_{\theta,p}$ and $(X,Y)_\theta$ are in $K_\theta(X,Y) \cap J_\theta(X,Y)$ for $0 < \theta < 1$ and $1 \leq p \leq \infty$. The Reiteration Theorem yields

$$((X,Y)_{\theta_0,q_0}, (X,Y)_{\theta_1,q_1})_{\theta,p} = (X,Y)_{(1-\theta)\theta_0+\theta\theta_1,p},$$

$$((X,Y)_{\theta_0}, (X,Y)_{\theta_1,q})_{\theta,p} = (X,Y)_{(1-\theta)\theta_0+\theta\theta_1,p},$$

$$((X,Y)_{\theta_0,q}, (X,Y)_{\theta_1})_{\theta,p} = (X,Y)_{(1-\theta)\theta_0+\theta\theta_1,p},$$

for $0<\theta_0,\theta_1<1$, $1\le p,\ q\le\infty$. Moreover, since X belongs to $K_0(X,Y)\cap J_0(X,Y)$, and Y belongs to $K_1(X,Y)\cap J_1(X,Y)$ between X and Y, then

$$((X,Y)_{\theta_0,q},Y)_{\theta,p}=(X,Y)_{(1-\theta)\theta_0+\theta,p},\quad ((X,Y)_{\theta_0},Y)_\theta=(X,Y)_{(1-\theta)\theta_0+\theta},$$

and

$$(X,(X,Y)_{\theta_1,q})_{\theta,p}=(X,Y)_{\theta_1\theta,p},\quad (X,(X,Y)_{\theta_1})_\theta=(X,Y)_{\theta_1\theta},$$

for $0<\theta_0,\theta_1<1$, $1\le p,\ q\le\infty$.

1.3.1 Examples

The following examples are immediate consequences of examples 1.1.8, 1.1.10, 1.1.9 and remark 1.3.7.

Example 1.3.8 *Let $0\le\theta_1<\theta_2\le 1$, $0<\theta<1$. Then*

$$(C_b^{\theta_1}(\mathbb{R}^n),C_b^{\theta_2}(\mathbb{R}^n))_{\theta,\infty}=C_b^{(1-\theta)\theta_1+\theta\theta_2}(\mathbb{R}^n).$$

If Ω is an open set in $\mathbb{R}^n$ with uniformly C^1 boundary, then

$$(C_b^{\theta_1}(\overline{\Omega}),C_b^{\theta_2}(\overline{\Omega}))_{\theta,\infty}=C_b^{(1-\theta)\theta_1+\theta\theta_2}(\overline{\Omega}).$$

Example 1.3.9 *Let $0\le\theta_1<\theta_2\le 1$, $0<\theta<1$, $1\le p<\infty$. Then*

$$(W^{\theta_1,p}(\mathbb{R}^n),W^{\theta_2,p}(\mathbb{R}^n))_{\theta,p}=W^{(1-\theta)\theta_1+\theta\theta_2,p}(\mathbb{R}^n).$$

If Ω is an open set in $\mathbb{R}^n$ with uniformly C^1 boundary, then

$$(W^{\theta_1,p}(\Omega),W^{\theta_2,p}(\Omega))_{\theta,\infty}=W^{(1-\theta)\theta_1+\theta\theta_2,p}(\Omega).$$

Example 1.3.10 *Let (Ω,μ) be a σ-finite measure space, and fix $1\le p_0,\ p_1$. Then for each $q\le\infty$, $0<\theta<1$ we have*

$$(L^{p_0,q_0}(\Omega),L^{p_1,q_1}(\Omega))_{\theta,q}=L^{p,q}(\Omega),$$

with

$$\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}.$$

Recalling that $L^{p,p}(\Omega)=L^p(\Omega)$, and taking $p_0=q_0$, $p_1=q_1$, we get

$$(L^{p_0}(\Omega),L^{p_1}(\Omega))_{\theta,q}=L^{p,q}(\Omega),\quad \frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}.$$

In particular,

$$(L^{p_0}(\Omega),L^{p_1}(\Omega))_{\theta,q}=L^{q,q}(\Omega)=L^q(\Omega)\ \text{ for }\theta=\left(1-\frac{p_0}{q}\right)\left(1-\frac{p_0}{p_1}\right)^{-1}.$$

Another generalization of example 1.1.8 is the following.

Example 1.3.11 *For $0<\theta<1$, $1\le p,\ q<\infty$, $m\in\mathbb{N}$,*

$$(L^p(\mathbb{R}^n),W^{m,p}(\mathbb{R}^n))_{\theta,q}=B^{m\theta}_{p,q}(\mathbb{R}^n).$$

Here $B^s_{p,q}(\mathbb{R}^n)$ is the Besov space defined as follows: if s is not an integer, let $[s]$ and $\{s\}$ be the integral and the fractional parts of s, respectively. Then $B^s_{p,q}(\mathbb{R}^n)$ consists of the functions $f \in W^{[s],p}(\mathbb{R}^n)$ such that

$$[f]_{B^s_{p,q}} =$$

$$\sum_{|\alpha|=[s]} \left(\int_{\mathbb{R}^n} \frac{dh}{|h|^{n+\{s\}q}} \left(\int_{\mathbb{R}^n} |D^\alpha f(x+h) - D^\alpha f(x)|^p dx \right)^{q/p} \right)^{1/q}$$

is finite. In particular, for $p = q$ we have $B^s_{p,p}(\mathbb{R}^n) = W^{s,p}(\mathbb{R}^n)$.

If $s = k \in \mathbb{N}$, then $B^k_{p,q}(\mathbb{R}^n)$ consists of the functions $f \in W^{k-1,p}(\mathbb{R}^n)$ such that

$$[f]_{B^k_{p,q}} = \sum_{|\alpha|=[s]-1} \left(\int_{\mathbb{R}^n} \frac{dh}{|h|^{n+q}} \right.$$

$$\left. \left(\int_{\mathbb{R}^n} |D^\alpha f(x+2h) - 2D^\alpha f(x+h) + D^\alpha f(x)|^p dx \right)^{q/p} \right)^{1/q}$$

is finite. For $m = 1$ see exercise 5, §1.1.2. For the complete proof see [36, §2.3, 2.4].

1.3.2 Applications. The theorems of Marcinkiewicz and Stampacchia

Let (Ω, μ), (Λ, ν) be two σ-finite measure spaces. Traditionally, a linear operator $T : L^1(\Omega) + L^\infty(\Omega) \mapsto L^1(\Lambda) + L^\infty(\Lambda)$ is said to be *of weak type* (p, q) if there is $M > 0$ such that

$$\sup_{\sigma>0} \sigma(\nu\{y \in \Lambda : |Tf(y)| > \sigma\})^{1/q} \le M\|f\|_{L^p(\Omega)},$$

for all $f \in L^p(\Omega)$. This is equivalent to say that the restriction of T to $L^p(\Omega)$ is a bounded operator from $L^p(\Omega)$ to $L^{q,\infty}(\Lambda)$. Indeed, by the properties of the nonincreasing rearrangements,

$$\sup_{\sigma>0} \sigma(\nu\{y \in \Lambda : |g(y)| > \sigma\})^{1/q} = \sup_{t>0} t^{1/q} g^*(t) = \|g\|_{L^{q,\infty}}. \tag{1.28}$$

T is said to be *of strong type* (p, q) if its restriction to $L^p(\Omega)$ is a bounded operator from $L^p(\Omega)$ to $L^q(\Lambda)$.

Since $L^q(\Lambda) = L^{q,q}(\Lambda) \subset L^{q,\infty}(\Lambda)$, then any operator *of strong type* (p, q) is also *of weak type* (p, q).

Theorem 1.3.12 *Let $T : L^1(\Omega) + L^\infty(\Omega) \mapsto L^1(\Lambda) + L^\infty(\Lambda)$ be of weak type (p_0, q_0) and (p_1, q_1), with constants M_0, M_1 respectively, and*

$$1 \le p_0,\ p_1 \le \infty,\ \ 1 < q_0,\ q_1 \le \infty,$$

$$q_0 \ne q_1,\ \ p_0 \le q_0,\ p_1 \le q_1.$$

For every $\theta \in (0, 1)$ define p and q by

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},\ \ \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then T is of strong type (p, q), and there is C independent of θ such that

$$\|Tf\|_{L^q(\Lambda)} \le C M_0^{1-\theta} M_1^\theta \|f\|_{L^p(\Omega)},\ \ f \in L^p(\Omega).$$

Proof. For $i = 1, 2$, T is bounded from $L^{p_i}(\Omega)$ to $L^{q_i,\infty}(\Lambda)$, with norm not exceeding CM_i. By the interpolation theorem 1.1.6, T is bounded from $(L^{p_0}(\Omega), L^{p_1}(\Omega))_{\theta,p}$ to $(L^{q_0,\infty}(\Lambda), L^{q_1,\infty}(\Lambda))_{\theta,p}$, and

$$\|T\|_{(L^{p_0}(\Omega),L^{p_1}(\Omega))_{\theta,p},(L^{q_0,\infty}(\Lambda),L^{q_1,\infty}(\Lambda))_{\theta,p})} \le CM_0^{1-\theta}M_1^{\theta}.$$

On the other hand, we know from example 1.3.10 that

$$(L^{p_0}(\Omega), L^{p_1}(\Omega))_{\theta,p} = L^{p,p}(\Omega) = L^p(\Omega),$$

and

$$L^{q_i,\infty}(\Lambda) = (L^1(\Lambda), L^\infty(\Lambda))_{1-1/q_i,\infty}, \quad i = 1, 2$$

(it is here that we need $q_i > 1$: $L^{1,\infty}(\Lambda)$ is not a real interpolation space between $L^1(\Lambda)$ and $L^\infty(\Lambda)$), so that by the Reiteration Theorem

$$(L^{q_0,\infty}(\Lambda), L^{q_1,\infty}(\Lambda))_{\theta,p} = (L^1(\Lambda), L^\infty(\Lambda))_{(1-\theta)(1-1/q_0)+\theta(1-1/q_1),p}$$

$$= (L^1(\Lambda), L^\infty(\Lambda))_{1-1/q,p}.$$

The last space is nothing but $L^{q,p}(\Lambda)$, again by example 1.1.10. Since $p_0 \le q_0$ and $p_1 \le q_1$ then $p \le q$, so that $L^{q,p}(\Lambda) \subset L^{q,q}(\Lambda) = L^q(\Lambda)$. It follows that T is bounded from $L^p(\Omega)$ to $L^q(\Lambda)$, with norm not exceeding $C'M_0^{1-\theta}M_1^{\theta}$. □

Theorem 1.3.12 is slightly less general than the complete Marcinkiewicz Theorem, which holds also for q_0 or $q_1 = 1$.

Since every T of strong type (p, q) is also of weak type (p, q) we may recover a part of the Riesz–Thorin Theorem from theorem 1.3.12: we get that if T is of strong type (p_i, q_i), $i = 0, 1$, with p_i, q_i subject to the restrictions in theorem 1.3.12, then T is of strong type (p, q). The full Riesz–Thorin Theorem will be proved in Chapter 2.

Let f be a locally integrable function. For each measurable subset $A \subset \Omega$ with positive measure $\mu(A)$ we define the mean value of f on A by

$$f_A = \frac{1}{\mu(A)} \int_A f(x)\mu(dx).$$

The space $BMO(\Omega)$ (BMO stands for bounded mean oscillation) consists of those locally integrable functions f such that

$$\sup_{0<\mu(A)<\infty} \frac{1}{\mu(A)} \int_A |f(x) - f_A|\mu(dx) < \infty.$$

If $\mu(\Omega) < \infty$ we see immediately that $L^\infty(\Omega) \subset BMO(\Omega) \subset L^1(\Omega)$.

It is possible to show that if $\Omega \subset \mathbb{R}^n$ is a Lipschitz bounded domain, then $BMO(\Omega)$ is contained in each $L^p(\Omega)$, $1 \le p < \infty$, and that the norms

$$\|f\|_{L^p} + [f]_{BMO,p} = \|f\|_{L^p} + \sup_A \left(\frac{1}{\mu(A)} \int_A |f(x) - f_A|^p dx \right)^{1/p}, \quad 1 \le p < \infty$$

are equivalent in $BMO(\Omega)$. This was proved by John and Nirenberg in the well known paper [24] in the case where Ω is a cube. For such Ω's we may extend the result of example 1.1.10 to the interpolation couple $(L^1(\Omega), BMO(\Omega))$.

Example 1.3.13 *Let Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz continuous boundary. Then for $0<\theta<1$, $1\le q\le\infty$,*

$$(L^1(\Omega), BMO(\Omega))_{\theta,q} = L^{\frac{1}{1-\theta},q}(\Omega).$$

As a consequence we get the main part of the Stampacchia interpolation theorem.

Theorem 1.3.14 *Let Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz continuous boundary, let $1\le r<\infty$ and let $T\in L(L^r(\Omega))\cap L(L^\infty(\Omega), BMO(\Omega))$, or else $T\in L(L^r(\Omega))\cap L(BMO(\Omega))$. Then $T\in L(L^p(\Omega))$ for every $p\in(r,\infty)$, and*

$$\|T\|_{L(L^p)} \le C\|T\|_{L(L^r)}^{r/p}\|T\|_{L(L^\infty,BMO)}^{1-r/p}$$

in the first case,

$$\|T\|_{L(L^p)} \le C\|T\|_{L(L^r)}^{r/p}\|T\|_{L(BMO)}^{1-r/p}$$

in the second case.

Proof. In the first case, the interpolation theorem 1.1.6 implies that $T\in L((L^r,L^\infty)_{\theta,p}, (L^r,BMO)_{\theta,p})$ for every $\theta\in(0,1)$ and $p\in[1,\infty]$, and

$$\|T\|_{L((L^r,L^\infty)_{\theta,p},(L^r,BMO)_{\theta,p})} \le \|T\|_{L(L^r)}^{1-\theta}\|T\|_{L(L^\infty,BMO)}^{\theta}.$$

By example 1.3.10, $(L^r(\Omega),L^\infty(\Omega))_{\theta,p} = L^{\frac{r}{1-\theta},p}(\Omega)$. By example 1.3.13 and reiteration, we still have

$$(L^r(\Omega), BMO(\Omega))_{\theta,p} = L^{\frac{r}{1-\theta},p}(\Omega).$$

Taking $\theta = 1-r/p$ (so that $r/(1-\theta)=p$) gives the first statement through the equality $L^{p,p}(\Omega)=L^p(\Omega)$. The proof of the second statement is similar. □

Campanato and Stampacchia ([15]) used the above interpolation theorem to prove optimal regularity results for elliptic boundary value problems, as follows.

Let

$$\mathcal{A} = \sum_{i,j=1}^n D_i(a_{ij}(x)D_j)$$

be an elliptic operator with L^∞ coefficients in a bounded open set $\Omega\subset\mathbb{R}^n$ with regular boundary. If f_{ji}, $j=0,\dots,n$ are in $L^2(\Omega)$, a weak solution of the Dirichlet problem

$$\begin{cases} \mathcal{A}u = f \ \text{ in } \Omega, \\ \\ u = 0 \ \text{ in } \partial\Omega \end{cases}$$

is any $u\in H_0^1(\Omega)$ such that for every $\varphi\in C_0^\infty(\Omega)$ it holds

$$\int_\Omega \sum_{i,j=1}^n a_{ij}(x)D_ju(x)D_i\varphi(x)dx = -\int_\Omega f_0(x)\varphi(x)dx + \int_\Omega \sum_{j=1}^n f_j(x)D_j\varphi(x)dx.$$

Using the Lax-Milgram theorem, it is not hard to see that the Dirichlet problem has a unique weak solution u, and $\|u\|_{H^1}\le C\sum_{j=0}^n \|f_j\|_{L^2}$. This is a first step in several basic courses in PDE's; see e.g. [8, Ch. IX] or [1]. The second step is a regularity theorem: if $f_j=0$ for each j and the coefficients $a_{ij}\in C^1(\overline{\Omega})$, then $u\in H^2(\Omega)$ and $\|u\|_{H^2}\le C\|f_0\|_{L^2}$. See again [8] or [1].

Moreover Campanato in [14, Thm. 16.II] was able to prove the following:

(i) if the coefficients a_{ij} are in $C^\alpha(\overline{\Omega})$ for some $\alpha \in (0,1)$ and the functions f_j are in $BMO(\Omega)$, then each derivative $D_i u$ belongs to $BMO(\Omega)$, and $\|D_i u\|_{BMO,2} \leq C\sum_{j=0}^n \|f_j\|_{BMO,2}$;

(ii) if the coefficients a_{ij} are in $C^{1+\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$, $f_j = 0$ and $f_0 \in BMO(\Omega)$, then $u \in H^2(\Omega)$ satisfies the equation a.e., each second order derivative $D_{ij}u$ belongs to $BMO(\Omega)$, and $\|D_{ij}u\|_{BMO,2} \leq C\|f_0\|_{BMO,2}$.

In case (i), applying theorem 1.3.14 with $r = 2$ to the operators T_i, $i = 1, \ldots, n$, defined by $T_i(f_0, \ldots, f_n) = D_i u$, u being the solution of the Dirichlet problem, we get that if the f_j's are in $L^p(\Omega)$, $2 < p < \infty$, then each derivative $D_i u$ belongs to $L^p(\Omega)$, and $\|D_i u\|_{L^p} \leq C\sum_{j=0}^n \|f_j\|_{L^p}$.

In case (ii), applying again theorem 1.3.14 with $r = 2$ to the operators T_{ij}, $i, j = 1, \ldots, n$, defined by $T_{ij}f_0 = D_i u$, we get that if $f_0 \in L^p(\Omega)$, $2 < p < \infty$, then each derivative $D_{ij}u$ belongs to $L^p(\Omega)$, and $\|D_{ij}u\|_{L^p} \leq C\|f_0\|_{L^p}$.

1.3.3 Exercises

1) Show that for $0 < \theta < 1$, $\theta \neq 1/2$,

$$(C_b(\mathbb{R}^n), C_b^2(\mathbb{R}^n))_{\theta,\infty} = C_b^{2\theta}(\mathbb{R}^n).$$

Hint: prove that $(C_b^1(\mathbb{R}^n), C_b^2(\mathbb{R}^n))_{\alpha,\infty} = C_b^{1+\alpha}(\mathbb{R}^n)$ using example 1.1.8, then use the Reiteration Theorem with $E = C_b^1(\mathbb{R}^n)$.

2) (extreme cases in reiteration)

(a) Using the examples of §1.1.1 find some interpolation couple (X, Y) and intermediate spaces in the classes J_0, J_1, K_1 between X and Y that do not coincide with X or Y.

(b) Give a direct proof of statement (ii) of the Reiteration Theorem in the case $(\theta_0, \theta_1) = (0, 1)$.

Chapter 2

Complex interpolation

The complex interpolation method is due to Calderon [13]. It works in complex interpolation couples. It may sound "artificial" compared to the more "natural" real interpolation method of chapter 1, see next definitions 2.1.1 and 2.1.3. It is in fact an abstraction and a generalization of the method used in the proof of the Riesz–Thorin interpolation theorem, which we show below.

The theorem of Riesz–Thorin. *Let (Ω,μ), (Λ,ν) be σ-finite measure spaces. Let $1 \le p_0, p_1, q_0, q_1 \le \infty$ and let $T : L^{p_0}(\Omega)+L^{p_1}(\Omega) \mapsto L^{q_0}(\Lambda)+L^{q_1}(\Lambda)$ be a linear operator such that*

$$T \in L(L^{p_0}(\Omega), L^{q_0}(\Lambda)) \cap L(L^{p_1}(\Omega), L^{q_1}(\Lambda)).$$

Then

$$T \in L(L^{p_\theta}(\Omega), L^{q_\theta}(\Lambda)), \quad 0 < \theta < 1,$$

with

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \tag{2.1}$$

and setting $M_i = \|T\|_{L(L^{p_i}(\Omega), L^{q_i}(\Lambda))}$, $i=0,1$, then

$$\|T\|_{L(L^{p_\theta}(\Omega), L^{q_\theta}(\Lambda))} \le M_0^{1-\theta} M_1^{\theta}.$$

In the case that some of the p_i's or the q_i's is ∞ the statement still holds if we set as usual $1/\infty = 0$.

Proof — We recall that the set of the simple functions (= finite linear combinations of characteristic functions of measurable sets with finite measure) $a : \Omega \mapsto \mathbb{C}$ is dense in $L^p(\Omega)$ for every $p \in [1,+\infty)$, and the set of the simple functions $b : \Lambda \mapsto \mathbb{C}$ is dense in $L^q(\Lambda)$, for every $q \in [1,+\infty)$. Moreover, for each measurable function $f : \Lambda \mapsto \mathbb{C}$ we have

$$\|f\|_{L^q(\Lambda)} = \sup \frac{1}{\|b\|_{L^{q'}(\Lambda)}} \left| \int_\Lambda f(x)b(x)\,\nu(dx) \right|$$

where $b \not\equiv 0$ varies in the set of the simple functions from Λ to $\mathbb{C}$.

Let $a : \Omega \mapsto \mathbb{C}$ be a simple function. Then $a \in L^p(\Omega)$ for each $p \in [1,+\infty]$; in particular $a \in L^{p_\theta}(\Omega)$. To estimate $\|Ta\|L^{q_\theta}(\Lambda)$ we use the above characterization of the L^{q_θ}-norm, i.e. we estimate the integral

$$\left| \int_\Lambda (Ta)(x)b(x)\,\nu(dx) \right|$$

where $b : \Lambda \mapsto \mathbb{C}$ is any simple function. To this aim, for every $z \in S = \{z = x + iy \in \mathbb{C} : 0 \le x \le 1\}$, we define

$$f(z)(x) = \begin{cases} |a(x)|^{p_\theta\left(\frac{1-z}{p_0}+\frac{z}{p_1}\right)} \frac{a(x)}{|a(x)|}, & \text{if } x \in \Omega,\ a(x) \neq 0, \\ \\ 0, & \text{if } x \in \Omega,\ a(x) = 0, \end{cases}$$

$$g(z)(x) = \begin{cases} |b(x)|^{q'_\theta\left(\frac{1-z}{q'_0}+\frac{z}{q'_1}\right)} \frac{b(x)}{|b(x)|}, & \text{if } x \in \Lambda,\ b(x) \neq 0, \\ \\ 0, & \text{if } x \in \Lambda,\ b(x) = 0. \end{cases}$$

Then $f(\theta) = a$, $g(\theta) = b$ and for each $z \in S$, $f(z) \in L^p(\Omega)$ for every p, $g(z) \in L^q(\Lambda)$ for every q. In particular, $f(z) \in L^{p_0}(\Omega) \cap L^{p_1}(\Omega)$ so that $Tf(z) \in L^{q_0}(\Lambda) \cap L^{q_1}(\Lambda)$, and the function

$$F : S \mapsto \mathbb{C}, \quad F(z) = \int_\Lambda Tf(z)\, g(z)\, \nu(dx)$$

is well defined, holomorphic in the interior of S, continuous and bounded in S, and $F(\theta) = \int_\Lambda (Ta)(x)b(x)\, \nu(dx)$ is the integral that we want to estimate. For every $t \in \mathbb{R}$ we have

$$|F(it)| \le \|Tf(it)\|_{L^{q_0}(\Lambda)} \|g(it)\|_{L^{q'_0}(\Lambda)} \le \|T\|_{L(L^{p_0}(\Omega), L^{q_0}(\Lambda))} \|a\|^{p_\theta/p_0}_{L^{p_\theta}(\Omega)} \|b\|^{q'_\theta/q'_0}_{L^{q'_\theta}(\Lambda)},$$

$$\begin{aligned} |F(1+it)| &\le \|Tf(1+it)\|_{L^{q_1}(\Lambda)} \|g(1+it)\|_{L^{q'_1}(\Lambda)} \\ &\le \|T\|_{L(L^{p_1}(\Omega), L^{q_1}(\Lambda))} \|a\|^{p_\theta/p_1}_{L^{p_\theta}(\Omega)} \|b\|^{q'_\theta/q'_1}_{L^{q'_\theta}(\Lambda)}. \end{aligned}$$

By the three lines theorem (see exercise 2, §2.1.3) we get

$$\begin{aligned} |F(\theta)| &= \left| \int_\Lambda Ta\, b\, \nu(dx) \right| \le (\sup_{t\in\mathbb{R}} |F(it)|)^{1-\theta} (\sup_{t\in\mathbb{R}} |F(1+it)|)^\theta \\ &\le \|T\|^{1-\theta}_{L(L^{p_0}(\Omega), L^{q_0}(\Lambda))} \|T\|^{\theta}_{L(L^{p_1}(\Omega), L^{q_1}(\Lambda))} \|a\|_{L^{p_\theta}(\Omega)} \|b\|_{L^{q'_\theta}(\Lambda)}. \end{aligned}$$

Since $\|Ta\|_{L^{q_\theta}(\Lambda)}$ is the supremum of $|F(\theta)|/\|b\|_{L^{q'_\theta}(\Lambda)}$ when $b \not\equiv 0$ runs in the set of the simple functions on Λ, we get

$$\|Ta\|_{L^{q_\theta}(\Lambda)} \le \|T\|^{1-\theta}_{L(L^{p_0}(\Omega), L^{q_0}(\Lambda))} \|T\|^{\theta}_{L(L^{p_1}(\Omega), L^{q_1}(\Lambda))} \|a\|_{L^{p_\theta}(\Omega)},$$

for every simple function $a : \Omega \mapsto \mathbb{C}$. Since the set of such a's is dense in $L^{p_\theta}(\Omega)$ the statement follows. □

Taking in particular $p_i = q_i$, we get that if $T \in L(L^{p_0}(\Omega), L^{p_0}(\Lambda)) \cap L(L^{p_1}(\Omega), L^{p_1}(\Lambda))$ then $T \in L(L^r(\Omega), L^r(\Lambda))$ for every $r \in [p_0, p_1]$.

The crucial part of the proof is the use of the three lines theorem for the function F. The explicit expression of F is not important; what is important is that F is holomorphic in the interior of S, continuous and bounded in S, that $F(\theta)$ leads to the norm $\|Ta\|_{L^{q_\theta}(\Lambda)}$, and that the behavior of F in $i\mathbb{R}$ and in $1+i\mathbb{R}$ is controlled. Banach space valued functions of this type are precisely those used in the construction of the complex interpolation spaces.

2.1 Definitions and properties

Throughout the section we shall use the maximum principle for holomorphic functions with values in a complex Banach space X: if Ω is a bounded open subset of $\mathbb{C}$ and $f:\overline{\Omega}\mapsto X$ is holomorphic in Ω and continuous in $\overline{\Omega}$, then $\|f(\zeta)\|_X \le \max\{\|f(z)\|_X : z\in\partial\Omega\}$, for every $\zeta\in\overline{\Omega}$. This is well known if $X=\mathbb{C}$, and may be recovered for general X by the following argument. For every $\zeta\in\overline{\Omega}$ let $x'\in X'$ be such that $\|f(\zeta)\|_X=\langle f(\zeta),x'\rangle$ and $\|x'\|_{X'}=1$. Applying the maximum principle to the complex function $z\mapsto\langle f(z),x'\rangle$ we get

$$\begin{aligned}\|f(\zeta)\|_X &= |\langle f(\zeta),x'\rangle| \le \max\{|\langle f(z),x'\rangle| : z\in\partial\Omega\}\\ &\le \max\{\|f(z)\|_X : z\in\partial\Omega\}.\end{aligned}$$

The maximum principle holds also for functions defined in strips. Dealing with complex interpolation, we shall consider the strip

$$S=\{z=x+iy\in\mathbb{C} : 0\le x\le 1\}.$$

If $f:S\mapsto X$ is holomorphic in the interior of S, continuous and bounded in S, then for each $\zeta\in S$

$$\|f(\zeta)\|_X\le\max\{\sup_{t\in\mathbb{R}}\|f(it)\|_X,\ \sup_{t\in\mathbb{R}}\|f(1+it)\|_X\}.$$

See exercise 1, §2.1.3.

Let (X,Y) be an interpolation couple of complex Banach spaces.

Definition 2.1.1 *Let S be the strip $\{z=x+iy\in\mathbb{C} : 0\le x\le 1\}$. $\mathcal{F}(X,Y)$ is the space of all functions $f:S\mapsto X+Y$ such that*

(i) f is holomorphic in the interior of the strip and continuous and bounded up to its boundary, with values in $X+Y$;

(ii) $t\mapsto f(it)\in C_b(\mathbb{R};X)$, $t\mapsto f(1+it)\in C_b(\mathbb{R};Y)$, and

$$\|f\|_{\mathcal{F}(X,Y)}=\max\{\sup_{t\in\mathbb{R}}\|f(it)\|_X,\sup_{t\in\mathbb{R}}\|f(1+it)\|_Y\}<\infty.$$

$\mathcal{F}_0(X,Y)$ is the subspace of $\mathcal{F}(X,Y)$ consisting of the functions $f:S\mapsto X+Y$ such that

$$\lim_{|t|\to\infty}\|f(it)\|_X=0,\quad \lim_{|t|\to\infty}\|f(1+it)\|_Y=0.$$

It is not hard to see that $\mathcal{F}(X,Y)$ and $\mathcal{F}_0(X,Y)$ are Banach spaces. Indeed, if f_n is a Cauchy sequence, the maximum principle gives, for all $z\in S$,

$$\begin{aligned}&\|f_n(z)-f_m(z)\|_{X+Y}\\ &\le\max\{\sup_{t\in\mathbb{R}}\|f_n(it)-f_m(it)\|_{X+Y},\sup_{t\in\mathbb{R}}\|f_n(1+it)-f_m(1+it)\|_{X+Y}\}\\ &\le\max\{\sup_{t\in\mathbb{R}}\|f_n(it)-f_m(it)\|_X,\sup_{t\in\mathbb{R}}\|f_n(1+it)-f_m(1+it)\|_Y\}.\end{aligned}$$

Therefore for every $z\in S$ there exists $f(z)=\lim_{n\to\infty}f_n(z)$ in $X+Y$, and it is easy to see that $f\in\mathcal{F}(X,Y)$. Since $t\mapsto f_n(it)$ converges in $C_b(\mathbb{R};X)$ and $t\mapsto f_n(1+it)$ converges in $C_b(\mathbb{R};Y)$, then f_n converges to f in $\mathcal{F}(X,Y)$. Moreover, since $\mathcal{F}_0(X,Y)$ is closed in $\mathcal{F}(X,Y)$, then $\mathcal{F}_0(X,Y)$ is a Banach space too.

An important technical lemma about the space $\mathcal{F}_0(X,Y)$ is the following one. Its intricate proof is due to Calderon [13, p. 132-133]. A very detailed proof is in the book of Krein–Petunin–Semenov [28, p. 217-220].

Lemma 2.1.2 *The linear hull of the functions* $e^{\delta z^2+\lambda z}a$, $\delta > 0$, $\lambda \in \mathbb{R}$, $a \in X \cap Y$, *is dense in* $\mathcal{F}_0(X,Y)$.

The complex interpolation spaces $[X,Y]_\theta$ are defined through the traces of the functions in $\mathcal{F}(X,Y)$.

Definition 2.1.3 *For every* $\theta \in [0,1]$ *set*

$$[X,Y]_\theta = \{f(\theta) :\ f \in \mathcal{F}(X,Y)\}, \quad \|a\|_{[X,Y]_\theta} = \inf_{f\in\mathcal{F}(X,Y),\, f(\theta)=a} \|f\|_{\mathcal{F}(X,Y)}.$$

$[X,Y]_\theta$ is isomorphic to the quotient space $\mathcal{F}(X,Y)/\mathcal{N}_\theta$, where $\mathcal{N}_\theta$ is the subset of $\mathcal{F}(X,Y)$ consisting of the functions which vanish at $z=\theta$. Since $\mathcal{N}_\theta$ is closed, the quotient space is a Banach space and so is $[X,Y]_\theta$.

Some immediate consequences of the definition are listed below.

(i) For every $\theta \in (0,1)$,
$$[X,Y]_\theta = [Y,X]_{1-\theta}.$$

(ii) We get an equivalent definition of $[X,Y]_\theta$ replacing the space $\mathcal{F}(X,Y)$ by the space $\mathcal{F}_0(X,Y)$. Indeed, for each $f \in \mathcal{F}(X,Y)$ and $\delta > 0$ the function $f_\delta(z) = e^{\delta(z-\theta)^2}f(z)$ is in $\mathcal{F}_0(X,Y)$, $f_\delta(\theta) = f(\theta)$, and $\|f_\delta\|_{\mathcal{F}(X,Y)} \le \max\{e^{\delta\theta^2},\ e^{\delta(1-\theta)^2}\}\|f\|_{\mathcal{F}(X,Y)}$. Letting $\delta \to 0$ we obtain also
$$\inf_{f\in\mathcal{F}(X,Y),\, f(\theta)=a} \|f\|_{\mathcal{F}(X,Y)} = \inf_{f\in\mathcal{F}_0(X,Y),\, f(\theta)=a} \|f\|_{\mathcal{F}(X,Y)}.$$

(iii) If $Y = X$ then $[X,X]_\theta = X$, with identical norms (see exercise 1, §2.1.3).

(iv) For every $t \in \mathbb{R}$ and for every $f \in \mathcal{F}(X,Y)$, then $f(\theta+it) \in [X,Y]_\theta$ for each $\theta \in (0,1)$, and $\|f(\theta+it)\|_{[X,Y]_\theta} = \|f(\theta)\|_{[X,Y]_\theta}$. (this is easily seen replacing the function f by $g(z) = f(z+it)$)

Finally, from lemma 2.1.2 it follows that $X \cap Y$ is dense in $[X,Y]_\theta$ for every $\theta \in (0,1)$. In the present chapter, this fact will be used only in example 2.1.11.

Proposition 2.1.4 *Let* $0 < \theta < 1$. *Then*

$$X \cap Y \subset [X,Y]_\theta \subset X + Y.$$

Proof — Let $a \in X \cap Y$. The constant function $f(z) = a$ belongs to $\mathcal{F}(X,Y)$, and

$$\|f\|_{\mathcal{F}(X,Y)} \le \max\{\|a\|_X, \|a\|_Y\}.$$

Therefore, $a = f(\theta) \in [X,Y]_\theta$ and $\|a\|_{[X,Y]_\theta} \le \|a\|_{X\cap Y}$.

The embedding $[X,Y]_\theta \subset X+Y$ follows again from the maximum principle: if $a = f(\theta)$ with $f \in \mathcal{F}(X,Y)$ then

$$\|a\|_{X+Y} \le \max\{\sup_{t\in\mathbb{R}} \|f(it)\|_{X+Y}, \sup_{t\in\mathbb{R}} \|f(1+it)\|_{X+Y}\}$$

$$\le \max\{\sup_{t\in\mathbb{R}} \|f(it)\|_{X}, \sup_{t\in\mathbb{R}} \|f(1+it)\|_{Y}\} = \|f\|_{\mathcal{F}(X,Y)}$$

so that $\|a\|_{X+Y} \le \|a\|_{[X,Y]_\theta}$. □

Remark 2.1.5 *Let $\mathcal{V}(X,Y)$ be the linear hull of the functions of the type $\varphi(z)x$, with $\varphi \in \mathcal{F}_0(\mathbb{C},\mathbb{C})$ and $x \in X \cap Y$. If $a \in X \cap Y$, its $[X,Y]_\theta$-norm may be obtained also as*

$$\|a\|_{[X,Y]_\theta} = \inf_{f\in\mathcal{V}(X,Y),\, f(\theta)=a} \|f\|_{\mathcal{F}(X,Y)}. \tag{2.2}$$

Proof — For each $\varepsilon > 0$ let $f_0 \in \mathcal{F}_0(X,Y)$ be such that $f_0(\theta) = a$ and $\|f_0\|_{\mathcal{F}(X,Y)} \le \|a\|_{[X,Y]_\theta} + \varepsilon$. Let $z \mapsto r(z)$ be a function continuous in S and holomorphic in the interior of S with values in the unit disk, and such that $r(\theta) = 0$, $r'(\theta) \ne 0$, $r(z) \ne 0$ for $z \ne \theta$. For instance, we may take

$$r(z) = \frac{z-\theta}{z+\theta}, \quad z \in S.$$

Set

$$f_1(z) = \frac{f_0(z) - e^{(z-\theta)^2}a}{r(z)}, \quad z \in S.$$

Then $f_1 \in \mathcal{F}_0(X,Y)$, so that by lemma 2.1.2 there exists a function

$$f_2(z) = \sum_{k=1}^{n} \exp(\delta_k z^2 + \gamma_k z)x_k,$$

with $\delta_k > 0$, $\gamma_k \in \mathbb{R}$, $x_k \in X \cap Y$, such that $\|f_1 - f_2\|_{\mathcal{F}(X,Y)} \le \varepsilon$. Set

$$f(z) = e^{(z-\theta)^2}a + r(z)f_2(z), \quad z \in S.$$

Then $f \in \mathcal{V}(X,Y)$ and

$$\|f\|_{\mathcal{F}(X,Y)} \le \|f_0\|_{\mathcal{F}(X,Y)} + \|f - f_0\|_{\mathcal{F}(X,Y)} \le \|a\|_{[X,Y]_\theta} + 2\varepsilon.$$

Remark 2.1.5 will be used in theorem 2.1.7 and in theorem 4.2.6.

We prove now that the spaces $[X,Y]_\theta$ are interpolation spaces.

Theorem 2.1.6 *Let (X_1,Y_1), (X_2,Y_2) be complex interpolation couples. If a linear operator T belongs to $L(X_1,X_2) \cap L(Y_1,Y_2)$, then the restriction of T to $[X_1,Y_1]_\theta$ belongs to $L([X_1,Y_1]_\theta,[X_2,Y_2]_\theta)$ for every $\theta \in (0,1)$. Moreover,*

$$\|T\|_{L([X_1,Y_1]_\theta,[X_2,Y_2]_\theta)} \le (\|T\|_{L[X_1,X_2]})^{1-\theta}(\|T\|_{L[Y_1,Y_2]})^{\theta}. \tag{2.3}$$

Proof — First let $\|T\|_{L(X_1,X_2)} \neq 0$ and $\|T\|_{L(Y_1,Y_2)} \neq 0$. If $a \in [X_1, Y_1]_\theta$, let $f \in \mathcal{F}(X_1, Y_1)$ be such that $f(\theta) = a$. Set

$$g(z) = \left(\frac{\|T\|_{L(X_1,X_2)}}{\|T\|_{L(Y_1,Y_2)}}\right)^{z-\theta} Tf(z), \quad z \in S.$$

Then $g \in \mathcal{F}(X_2, Y_2)$, and

$$\|g(it)\|_{X_2} \leq (\|T\|_{L(X_1,X_2)})^{1-\theta}(\|T\|_{L(Y_1,Y_2)})^{\theta}\|f(it)\|_{X_1},$$

$$\|g(1+it)\|_{Y_2} \leq (\|T\|_{L(X_1,X_2)})^{1-\theta}(\|T\|_{L(Y_1,Y_2)})^{\theta}\|f(1+it)\|_{Y_1},$$

so that $\|g\|_{\mathcal{F}(X_2,Y_2)} \leq (\|T\|_{L(X_1,X_2)})^{1-\theta}(\|T\|_{L(Y_1,Y_2)})^{\theta}\|f\|_{\mathcal{F}(X_1,Y_1)}$. Therefore $Ta = g(\theta) \in [X_2, Y_2]_\theta$, and

$$\|Ta\|_{[X_2,Y_2]_\theta} \leq \|g\|_{\mathcal{F}(X_2,Y_2)} \leq (\|T\|_{L(X_1,X_2)})^{1-\theta}(\|T\|_{L(Y_1,Y_2)})^{\theta}\|f\|_{\mathcal{F}(X_1,Y_1)}.$$

Taking the infimum over all $f \in \mathcal{F}(X_1, Y_1)$ we get

$$\|Ta\|_{[X_2,Y_2]_\theta} \leq (\|T\|_{L[X_1,X_2]})^{1-\theta}(\|T\|_{L[Y_1,Y_2]})^{\theta}\|a\|_{[X_1,Y_1]_\theta}.$$

If either $\|T\|_{L(X_1,X_2)}$ or $\|T\|_{L(Y_1,Y_2)}$ vanishes, replace it by $\varepsilon > 0$ in the definition of g and then let $\varepsilon \to 0$ to get the statement.

If both $\|T\|_{L(X_1,X_2)}$ and $\|T\|_{L(Y_1,Y_2)}$ vanish, set $g(z) = Tf(z)$. □

Theorem 2.1.6 has an interesting extension to linear operators depending on $z \in S$.

Theorem 2.1.7 *Let (X_1, Y_1), (X_2, Y_2) be complex interpolation couples.*

For every $z \in S$ let $T_z \in L(X_1 \cap Y_1, X_2 + Y_2)$ be such that $z \mapsto T_z x$ is holomorphic in S and continuous and bounded in S for every $x \in X_1 \cap Y_1$, with values in $X_2 + Y_2$. Moreover assume that $t \mapsto T_{it}x \in C(\mathbb{R}; L(X_1, X_2))$, $t \mapsto T_{1+it}x \in C(\mathbb{R}; L(Y_1, Y_2))$ and that $\|T_{it}\|_{L(X_1,X_2)}$, $\|T_{1+it}\|_{L(Y_1,Y_2)}$ are bounded by a constant independent of t.

Then, setting

$$M_0 = \sup_{t\in\mathbb{R}} \|T_{it}\|_{L(X_1,X_2)}, \quad M_1 = \sup_{t\in\mathbb{R}} \|T_{1+it}\|_{L(Y_1,Y_2)}$$

for every $\theta \in (0, 1)$ we have

$$\|T_\theta x\|_{[X_2,Y_2]_\theta} \leq M_0^{1-\theta} M_1^{\theta} \|x\|_{[X_1,Y_1]_\theta}$$

so that T_θ has an extension belonging to $L([X_1, Y_1]_\theta, [X_2, Y_2]_\theta)$ (which we still call T_θ) satisfying

$$\|T_\theta\|_{L([X_1,Y_1]_\theta,[X_2,Y_2]_\theta)} \leq M_0^{1-\theta} M_1^{\theta}. \tag{2.4}$$

Proof — The proof is just a modification of the proof of theorem 2.1.6.

Assume first that M_0 and M_1 are positive. For every $a \in X_1 \cap Y_1$ let $f \in \mathcal{V}(X_1, Y_1)$ be such that $f(\theta) = a$ (see remark 2.1.5), and set

$$g(z) = \left(\frac{M_0}{M_1}\right)^{z-\theta} T_z f(z), \quad z \in S.$$

Then $g \in \mathcal{F}(X_2, Y_2)$ and

$$\|g(it)\|_{X_2} \le M_0^{1-\theta} M_1^{\theta} \|f(it)\|_{X_1},$$

$$\|g(1+it)\|_{Y_2} \le M_0^{1-\theta} M_1^{\theta} \|f(1+it)\|_{Y_1},$$

so that $\|g\|_{\mathcal{F}(X_2,Y_2)} \le M_0^{1-\theta} M_1^{\theta} \|f\|_{\mathcal{F}(X_1,Y_1)}$. Therefore $T_\theta a = g(\theta) \in [X_2, Y_2]_\theta$, and

$$\|T_\theta a\|_{[X_2,Y_2]_\theta} \le M_0^{1-\theta} M_1^{\theta} \inf_{f \in \mathcal{V}(X_1,Y_1),\, f(\theta)=a} \|f\|_{\mathcal{F}(X_1,Y_1)},$$

but by remark 2.1.5,

$$\|a\|_{[X_1,Y_1]_\theta} = \inf_{f \in \mathcal{V}(X_1,Y_1),\, f(\theta)=a} \|f\|_{\mathcal{F}(X_1,Y_1)},$$

and the statement follows.

If either M_0 or M_1 vanishes, replace it by ε in the definition of g, and then let $\varepsilon \to 0$. If both M_0 and M_1 vanish, define g by $g(z) = T_z f(z)$ and follow the above arguments. □

Let us come back to theorem 2.1.6 and to its consequences. The same proof of corollary 1.1.7 (through the equality $[\mathbb{C}, \mathbb{C}]_\theta = \mathbb{C}$ with the same norm) yields

Corollary 2.1.8 *For every $\theta \in (0,1)$ we have*

$$\|y\|_{[X,Y]_\theta} \le \|y\|_X^{1-\theta} \|y\|_Y^{\theta}, \quad \forall y \in X \cap Y. \tag{2.5}$$

Therefore, $[X,Y]_\theta \in J_\theta(X,Y)$. This means that $(X,Y)_{\theta,1} \subset [X,Y]_\theta$, thanks to proposition 1.3.2. It is also true that $[X,Y]_\theta \subset (X,Y)_{\theta,\infty}$; to prove it we need the following lemma, which gives a Poisson formula for holomorphic functions in a strip with values in Banach spaces.

Lemma 2.1.9 *For every bounded $f : \overline{S} \mapsto X$ which is continuous in $\overline{S}$ and holomorphic in S we have*

$$f(z) = f_0(z) + f_1(z), \quad z = x + iy \in S,$$

where

$$\begin{aligned} f_0(z) &= \int_{\mathbb{R}} e^{\pi(y-t)} \sin(\pi x) \frac{f(it)}{\sin^2(\pi x) + (\cos(\pi x) - \exp(\pi(y-t)))^2}\, dt, \\ f_1(z) &= \int_{\mathbb{R}} e^{\pi(y-t)} \sin(\pi x) \frac{f(1+it)}{\sin^2(\pi x) + (\cos(\pi x) + \exp(\pi(y-t)))^2}\, dt. \end{aligned} \tag{2.6}$$

Sketch of the proof — Let first $X = \mathbb{C}$. Then (2.6) may be obtained using the Poisson formula for the unit circle,

$$\widetilde{f}(\zeta) = \frac{1}{2\pi} \int_{|\lambda|=1} \widetilde{f}(\lambda) \frac{1 - |\zeta|^2}{|\zeta - \lambda|^2}\, d\lambda, \quad |\zeta| < 1,$$

(which holds for every $\widetilde{f}$ which is holomorphic in the interior and continuous up to the boundary), and the conformal mapping

$$\zeta(z) = \frac{e^{\pi i z} - i}{e^{\pi i z} + i}, \quad z \in \overline{S},$$

which transforms S into the unit circle. If X is a general Banach space (2.6) follows as usual, considering the complex functions $z \mapsto \langle f(z), x' \rangle$ for every $x' \in X'$, and applying (2.6) to each of them. □

Proposition 2.1.10 *For every $\theta \in (0,1)$, $[X,Y]_\theta \in K_\theta(X,Y)$, that is $[X,Y]_\theta$ is continuously embedded in $(X,Y)_{\theta,\infty}$.*

Proof — Let $a \in [X,Y]_\theta$. For every $f \in \mathcal{F}(X,Y)$ such that $f(\theta) = a$ split $a = f_0(\theta)+f_1(\theta)$ according to (2.6).

Note that for $z = x+iy \in S$ we have

$$0 < \int_{\mathbb{R}} e^{\pi(y-t)} \sin(\pi x) \frac{1}{\sin^2(\pi x) + (\cos(\pi x) - \exp(\pi(y-t)))^2}\, dt < 1,$$

$$0 < \int_{\mathbb{R}} e^{\pi(y-t)} \sin(\pi x) \frac{1}{\sin^2(\pi x) + (\cos(\pi x) + \exp(\pi(y-t)))^2}\, dt < 1.$$

Indeed, both kernels are positive so that both integrals are positive; moreover if f is holomorphic in S, continuous and bounded in $\overline{S}$ and $f \equiv 1$ on $i\mathbb{R}$ and on $1+i\mathbb{R}$ then $f \equiv 1$ in S, so that the sum of the integrals is 1.

Therefore for every $z \in S$, $f_0(z) \in X$, $\|f_0(z)\|_X \le \sup_{\tau\in\mathbb{R}} \|f(i\tau)\|_X$, and $f_1(z) \in Y$, $\|f_1(z)\|_Y \le \sup_{\tau\in\mathbb{R}} \|f(1+i\tau)\|_Y$. Then for each $t > 0$ we get

$$K(t,a) \le \|f_0(\theta)\|_X + t\|f_1(\theta)\|_Y \le \sup_{\tau\in\mathbb{R}} \|f(i\tau)\|_X + t \sup_{\tau\in\mathbb{R}} \|f(1+i\tau)\|_Y.$$

The function $g(z) = t^{\theta-z} f(z)$ is in $\mathcal{F}(X,Y)$, and $g(\theta) = a$. Applying the above estimate to g we get

$$K(t,a) \le \sup_{\tau\in\mathbb{R}} \|f(i\tau)\|_X t^\theta + t \sup_{\tau\in\mathbb{R}} \|f(1+i\tau)\|_Y t^{\theta-1} \le 2t^\theta \|f\|_{\mathcal{F}(X,Y)}.$$

Since f is arbitrary,

$$K(t,a) \le 2t^\theta \|a\|_{[X,Y]_\theta},$$

and the statement follows. □

By corollary 2.1.8 and proposition 1.1.3, $[X,Y]_\theta \in J_\theta(X,Y) \cap K_\theta(X,Y)$.

This implies, through proposition 1.1.4, that $[X,Y]_{\theta_2} \subset [X,Y]_{\theta_1}$ for $\theta_1 < \theta_2$ whenever $Y \subset X$.

This also allows to use the Reiteration Theorem to characterize the real interpolation spaces between complex interpolation spaces. We get, for $0 < \theta_1 < \theta_2 < 1$, $0 < \theta < 1$, $1 \le p \le \infty$,

$$([X,Y]_{\theta_1}, [X,Y]_{\theta_2})_{\theta,p} = (X,Y)_{(1-\theta)\theta_1+\theta\theta_2,p}.$$

Further reiteration properties are the following. Calderon ([13]) showed that if one of the spaces X, Y is continuously embedded in the other one, or if X, Y are reflexive and $X \cap Y$ is dense both in X and in Y, then

$$[[X,Y]_{\theta_1}, [X,Y]_{\theta_2}]_\theta = [X,Y]_{(1-\theta)\theta_1+\theta\theta_2},$$

Lions ([30]) proved that if X and Y are reflexive, then for $0 < \theta_1 < \theta_2 < 1$, $0 < \theta < 1$, $1 < p < \infty$,

$$[(X,Y)_{\theta_1,p}, (X,Y)_{\theta_2,p}]_\theta = (X,Y)_{(1-\theta)\theta_1+\theta\theta_2,p}.$$

The question whether $[X,Y]_\theta$ coincides with some $(X,Y)_{\theta,p}$ has no general answer. We will see in the next chapter (sect. 3.4) that if X and Y are Hilbert spaces then

$$[X,Y]_\theta = (X,Y)_{\theta,2}, \;\; 0<\theta<1,$$

but in the non hilbertian case there are no general rules. See next examples 2.1.11 and 2.1.12.

2.1.1 Examples

Example 2.1.11 *Let (Ω,μ) be a measure space with σ-finite measure, and let $1 \le p_0,\, p_1 \le \infty$, $0<\theta<1$. Then*

$$[L^{p_0}(\Omega), L^{p_1}(\Omega)]_\theta = L^p(\Omega), \;\; \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},$$

and their norms coincide. (In the case $p_0=\infty$ or $p_1=\infty$ the statement is correct if we set as usual $1/\infty = 0$).

Proof — The proof follows the proof of the Riesz–Thorin theorem at the beginning of the chapter.

We recall that $L^{p_0}(\Omega)\cap L^{p_1}(\Omega)$ is dense both in $L^p(\Omega)$ and in $[L^{p_0}(\Omega), L^{p_1}(\Omega)]_\theta$.

Let $a \in L^{p_0}(\Omega)\cap L^{p_1}(\Omega)$. We may assume without loss of generality that $\|a\|_{L^p}=1$. For $z\in\overline{S}$ set

$$f(z)(x) = |a(x)|^{p\left(\frac{1-z}{p_0}+\frac{z}{p_1}\right)}\frac{a(x)}{|a(x)|}, \;\; \text{if } x\in\Omega, \; a(x)\neq 0,$$

$$f(z)(x) = 0, \;\; \text{if } x\in\Omega, \; a(x)=0.$$

Then f is continuous in S and holomorphic in the interior of S with values in $L^{p_0}(\Omega) + L^{p_1}(\Omega)$, and for each $t\in\mathbb{R}$

$$|f(it)(x)| = |a(x)|^{\frac{p}{p_0}}, \;\; \|f(it)\|_{L^{p_0}} \le \|a\|_{L^p}^{p/p_0} = 1,$$

$$|f(1+it)(x)| = |a(x)|^{\frac{p}{p_1}}, \;\; \|f(1+it)\|_{L^{p_1}} \le \|a\|_{L^p}^{p/p_1} = 1.$$

Moreover, $t\mapsto f(it)$ is continuous with values in $L^{p_0}(\Omega)$ and $t\mapsto f(1+it)$ is continuous with values in $L^{p_1}(\Omega)$. Therefore, $f\in\mathcal{F}(L^{p_0}(\Omega), L^{p_1}(\Omega))$ and $\|f\|_{\mathcal{F}(L^{p_0},L^{p_1})}\le 1$. Since $f(\theta)=a$, then

$$\|a\|_{[L^{p_0},L^{p_1}]_\theta} \le 1 = \|a\|_{L^p}.$$

To prove the opposite inequality we remark that

$$1 = \|a\|_{L^p} = \sup\left\{\left|\int_\Omega a(x)b(x)\mu(dx)\right| : \, b\in L^{p_0'}\cap L^{p_1'}, \, \|b\|_{L^{p'}}=1\right\}.$$

For every $b\in L^{p_0'}\cap L^{p_1'}$ with $\|b\|_{L^{p'}}=1$ set, as before

$$g(z)(x) = |b(x)|^{p'\left(\frac{1-z}{p_0'}+\frac{z}{p_1'}\right)}\frac{b(x)}{|b(x)|}, \;\; \text{if } x\in\Omega, \; b(x)\neq 0,$$

$$g(z)(x) = 0, \;\; \text{if } x\in\Omega, \; b(x)=0,$$

and define, for every $f \in \mathcal{F}(L^{p_0}, L^{p_1})$ such that $f(\theta) = a$,

$$F(z) = \int_\Omega f(z)(x)g(z)(x)dx, \quad z \in \overline{S}.$$

Then F is holomorphic in the interior of S and continuous in S, so that the maximum principle (see exercise 2, §2.1.3) implies that for every $z \in S$ it holds

$$|F(z)| \leq \max\{\sup_{t\in\mathbb{R}} |F(it)|, \sup_{t\in\mathbb{R}} |F(1+it)|\}.$$

But $|F(it)|$ and $|F(1+it)|$ may be easily estimated:

$$|F(it)| \leq \|f(it)\|_{L^{p_0}} \|g(it)\|_{L^{p_0'}} = \|f(it)\|_{L^{p_0}} \|b\|_{L^{p'}}^{p'/p_0'} = \|f(it)\|_{L^{p_0}},$$

$$|F(1+it)| \leq \|f(1+it)\|_{L^{p_1}} \|g(1+it)\|_{L^{p_1'}}$$

$$= \|f(1+it)\|_{L^{p_1}} \|b\|_{L^{p'}}^{p'/p_1'} = \|f(1+it)\|_{L^{p_1}},$$

so that

$$|F(z)| \leq \max\{\sup_{t\in\mathbb{R}} \|f(it)\|_{L^{p_0}}, \sup_{t\in\mathbb{R}} \|f(1+it)\|_{L^{p_1}}\}$$

$$\leq \|f\|_{\mathcal{F}(L^{p_0}, L^{p_1})}, \quad z \in S.$$

Therefore,

$$\left| \int_\Omega a(x)b(x)dx \right| = |F(\theta)| \leq \|f\|_{\mathcal{F}(L^{p_0}, L^{p_1})}.$$

Since b is arbitrary,

$$\|a\|_{L^p} \leq \|f\|_{\mathcal{F}(L^{p_0}, L^{p_1})}.$$

Since f is arbitrary,

$$\|a\|_{L^p} \leq \|a\|_{[L^{p_0}, L^{p_1}]_\theta}.$$

Therefore the identity is an isometry between $L^{p_0} \cap L^{p_1}$ with the L^p norm and $L^{p_0} \cap L^{p_1}$ with the $[L^{p_0}, L^{p_1}]_\theta$ norm. Since $L^{p_0} \cap L^{p_1}$ is dense respectively in L^p and in $[L^{p_0}, L^{p_1}]_\theta$, the statement follows. □

Example 2.1.12 *For* $0 < \theta < 1$, $1 < p < \infty$, $m \in \mathbb{N}$,

$$[L^p(\mathbb{R}^n), W^{m,p}(\mathbb{R}^n)]_\theta = H^{m\theta,p}(\mathbb{R}^n).$$

The proof is in [36, §2.4.2].

We recall that for $s > 0$

$$H^{s,p}(\mathbb{R}^n) = \{f \in L^p(\mathbb{R}^n) : \|f\|_{H^{s,p}} = \|\mathcal{F}^{-1}(1+|x|^2)^{s/2}\mathcal{F}f\|_{L^p} < \infty\},$$

where $\mathcal{F}$ is the Fourier transform. It is known that if $s = k$ is integer then

$$H^{k,p}(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n), \quad k \in \mathbb{N}.$$

Moreover it is known that

$$B^s_{p,p}(\mathbb{R}^n) \subset H^{s,p}(\mathbb{R}^n) \subset B^s_{p,2}(\mathbb{R}^n), \quad 1 < p \leq 2,$$

$$B^s_{p,2}(\mathbb{R}^n) \subset H^{s,p}(\mathbb{R}^n) \subset B^s_{p,p}(\mathbb{R}^n), \quad 2 \leq p < \infty,$$

and the inclusions are strict if $p \neq 2$. See [36, §2.3.3]. We recall (example 1.3.11) that $(L^p(\mathbb{R}^n), W^{m,p}(\mathbb{R}^n))_{\theta,p} = B^{m\theta}_{p,p}(\mathbb{R}^n)$. Therefore

$$[L^p(\mathbb{R}^n), W^{m,p}(\mathbb{R}^n)]_\theta \neq (L^p(\mathbb{R}^n), W^{m,p}(\mathbb{R}^n))_{\theta,p}$$

unless $p = 2$.

2.1.2 The theorems of Hausdorff–Young, Riesz–Thorin, Stein

Applying theorem 2.1.6 to the spaces $X_1 = L^{p_0}(\Omega)$, $X_2 = L^{q_0}(\Lambda)$, $Y_1 = L^{p_1}(\Omega)$, $Y_2 = L^{q_1}(\Lambda)$ and recalling example 2.1.11 we get the Riesz–Thorin theorem, as stated at the beginning of the chapter. However, the proof of example 2.1.11 is modeled on the proof of the Riesz–Thorin theorem, so that this has not to be considered an alternative proof.

An important application of the Riesz–Thorin theorem (or, equivalently, of theorem 2.1.6 and example 2.1.11) is the theorem of Hausdorff and Young on the Fourier transform in $L^p(\mathbb{R}^n)$. We set, for every $f \in L^1(\mathbb{R}^n)$,

$$(\mathcal{F}f)(k) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\langle x,k\rangle} f(x)dx, \quad k \in \mathbb{R}^n.$$

As easily seen, $\|\mathcal{F}f\|_{L^2} = \|f\|_{L^2}$ for every $f \in C_0^\infty(\mathbb{R}^n)$, so that $\mathcal{F}$ is canonically extended to an isometry (still denoted by $\mathcal{F}$) to $L^2(\mathbb{R}^n)$.

Theorem 2.1.13 *If $1 < p \leq 2$, $\mathcal{F}$ is a bounded operator from $L^p(\mathbb{R}^n)$ to $L^{p'}(\mathbb{R}^n)$, $p' = p/(p-1)$, and*

$$\|\mathcal{F}\|_{L(L^p,L^{p'})} \leq \frac{1}{(2\pi)^{n(1/p-1/2)}}.$$

Proof — Since $\|\mathcal{F}\|_{L(L^1,L^\infty)} \leq (2\pi)^{-n/2}$ and $\|\mathcal{F}\|_{L(L^2)} = 1$, by the Riesz–Thorin theorem $\mathcal{F} \in L(L^{p_\theta}, L^{q_\theta})$ for every $\theta \in (0,1)$, p_θ and q_θ being defined by (2.1): $p_\theta = 2/(1+\theta)$, $q_\theta = 2/(1-\theta) = p'_\theta$. Moreover,

$$\|\mathcal{F}\|_{L(L^{p_\theta},L^{p'_\theta})} \leq \|\mathcal{F}\|^\theta_{L(L^1,L^\infty)} \|\mathcal{F}\|^{1-\theta}_{L(L^2)} \leq \left(\frac{1}{(2\pi)^{n/2}}\right)^{2/p-1}. \tag{2.7}$$

The use of the Riesz–Thorin theorem may be avoided by using directly the results of theorem 2.1.6 and of example 2.1.11: by theorem 2.1.6 $\mathcal{F}$ is a bounded operator from $[L^1, L^2]_\theta$ to $[L^\infty, L^2]_\theta$ for every $\theta \in (0,1)$; by example 2.1.11, $[L^1, L^2]_\theta = L^{p_\theta}$, $p_\theta = 2/(1+\theta)$, and $[L^\infty, L^2]_\theta = L^{q_\theta}$, $q_\theta = 2/(1-\theta)$, with identical norms. Therefore, (2.7) holds.

When θ runs in $(0,1)$, $p_\theta = 2/(1+\theta)$ runs in $(1,2)$, and the statement is proved. □

A useful generalization of the Riesz-Thorin theorem is the Stein interpolation theorem. It is obtained by applying theorem 2.1.7 to the interpolation couples $(L^{p_0}(\Omega), L^{p_1}(\Omega))$, $(L^{q_0}(\Lambda), L^{q_1}(\Lambda))$, using the characterization of example 2.1.11.

Theorem 2.1.14 *Let (Ω,μ), (Λ,ν) be σ-finite measure spaces. Let p_0, p_1, q_0, $q_1 \in [1,\infty]$ and define as usual p_θ, q_θ by*

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad 0 < \theta < 1.$$

Assume that for every $z \in S$, $T_z : L^{p_0}(\Omega) \cap L^{p_1}(\Omega) \mapsto L^{q_0}(\Lambda) + L^{q_1}(\Lambda)$ is a linear operator such that

- *(i) for each $f \in L^{p_0}(\Omega) \cap L^{p_1}(\Omega)$, $z \mapsto T_z f$ is holomorphic in the interior of S and continuous and bounded in S with values in $L^{q_0}(\Lambda) + L^{q_1}(\Lambda)$;*

- *(ii) for each $f \in L^{p_0}(\Omega) \cap L^{p_1}(\Omega)$, $t \mapsto T_{it} f$ is continuous and bounded in $\mathbb{R}$ with values in $L^{q_0}(\Lambda)$, $t \mapsto T_{1+it} f$ is continuous and bounded in $\mathbb{R}$ with values in $L^{q_1}(\Lambda)$;*

(iii) there are M_0, $M_1 > 0$ such that for each $f \in L^{p_0}(\Omega) \cap L^{p_1}(\Omega)$,

$$\sup_{t\in\mathbb{R}} \|T(t)f\|_{L^{q_0}(\Lambda)} \le M_0 \|f\|_{L^{p_0}(\Omega)}, \quad \sup_{t\in\mathbb{R}} \|T(t)f\|_{L^{q_1}(\Lambda)} \le M_1 \|f\|_{L^{p_1}(\Omega)}.$$

Then for each $\theta \in (0,1)$ and for each $f \in L^{p_0}(\Omega) \cap L^{p_1}(\Omega)$ we have

$$\|T_\theta f\|_{L^{q_\theta}(\Lambda)} \le M_0^{1-\theta} M_1^\theta \|f\|_{L^{p_\theta}(\Omega)}.$$

Therefore, T_θ may be extended to a bounded operator (which we still call T_θ) from $L^{p_\theta}(\Omega)$ to $L^{q_\theta}(\Lambda)$, with p_θ and q_θ defined in (2.1), and

$$\|T_\theta\|_{L(L^{p_\theta}(\Omega), L^{q_\theta}(\Lambda))} \le M_0^{1-\theta} M_1^\theta.$$

Theorem 2.1.14 has a slightly sharper version, stated below, obtained modifying the direct proof of the Riesz–Thorin theorem. We recall that if (Ω, μ) is a measure space, a simple function is a (finite) linear combination of characteristic functions of measurable sets with finite measure.

Theorem 2.1.15 *Let (Ω, μ), (Λ, ν) be σ-finite measure spaces. Assume that for every $z \in S$, T_z is a linear operator defined in the set of the simple functions on Ω, with values into measurable functions on Λ, such that for every couple of simple functions $a : \Omega \mapsto \mathbb{C}$ and $b : \Lambda \mapsto \mathbb{C}$, the product $T_z a \cdot b$ is integrable on Λ and*

$$z \mapsto \int_\Lambda (T_z f)(x) g(x) \nu(dx),$$

is continuous and bounded in S, holomorphic in the interior of S.

Assume moreover that for some p_j, $q_j \in [1, +\infty]$, $j = 0, 1$, we have

$$\|T_{it} a\|_{L^{q_0}(\Lambda)} \le M_0 \|a\|_{L^{p_0}(\Omega)}, \quad \|T_{1+it} a\|_{L^{q_1}(\Lambda)} \le M_0 \|a\|_{L^{p_1}(\Omega)}, \quad t \in \mathbb{R},$$

for every simple function a. Then for each $\theta \in (0,1)$, T_θ may be extended to a bounded operator (which we still call T_θ) from $L^{p_\theta}(\Omega)$ to $L^{q_\theta}(\Lambda)$, with p_θ and q_θ defined in (2.1), and

$$\|T_\theta\|_{L(L^{p_\theta}(\Omega), L^{q_\theta}(\Lambda))} \le M_0^{1-\theta} M_1^\theta.$$

Proof — The proof is just a modification of the proof of the Riesz–Thorin theorem. For every couple of simple functions $a\Omega \mapsto \mathbb{C}$, $b : \Lambda \mapsto \mathbb{C}$, we apply the three lines theorem to the function

$$F(z) = \int_\Lambda T_z f(z)\, g(z)\, \nu(dx)$$

where f and g are defined as in the proof of the Riesz–Thorin theorem, i.e.

$$f(z)(x) = |a(x)|^{p\left(\frac{1-z}{p_0} + \frac{z}{p_1}\right)} \frac{a(x)}{|a(x)|}, \ \text{ if } x \in \Omega,\ a(x) \neq 0;$$

$$f(z)(x) = 0, \ \text{ if } x \in \Omega,\ a(x) = 0.$$

$$g(z)(x) = |b(x)|^{p'\left(\frac{1-z}{p_0'} + \frac{z}{p_1'}\right)} \frac{b(x)}{|b(x)|}, \ \text{ if } x \in \Lambda,\ b(x) \neq 0,$$

$$g(z)(x) = 0, \text{ if } x \in \Lambda,\ b(x) = 0.$$

We get

$$|F(\theta)| = \left| \int_\Lambda (T_\theta a)(x) b(x)\, \nu(dx) \right| \le M_0^{1-\theta} M_1^\theta \|a\|_{L^{p_\theta}(\Omega)} \|b\|_{L^{q'_\theta}(\Lambda)},$$

so that

$$\|Ta\|_{L^{q_\theta}(\Lambda)} \le M_0^{1-\theta} M_1^\theta \|a\|_{L^{p_\theta}(\Omega)},$$

for every simple a defined in Ω. Since the set of such a's is dense in $L^{p_\theta}(\Omega)$ the statement follows. □

2.1.3 Exercises

1) *The maximum principle for functions defined on a strip.* Let $f : S \mapsto X$ be holomorphic in the interior of S, continuous and bounded in S. Prove that for each $\zeta \in S$

$$\|f(\zeta)\| \le \max\{\sup_{t\in\mathbb{R}} \|f(it)\|,\ \sup_{t\in\mathbb{R}} \|f(1+it)\|\}.$$

(Hint: for each $\varepsilon \in (0, 1)$ let z_0 be such that $\|f(z_0)\| \ge \|f\|_\infty(1-\varepsilon)$; consider the functions $f_\delta(z) = \exp(\delta(z - z_0)^2)f(z)$, and apply the maximum principle in the rectangle $[0, 1] \times [-M, M]$ with M large).

2) *The three lines theorem.* Let $f : S \mapsto X$ be holomorphic the interior of S, continuous and bounded in S. Show that

$$\|f(\theta)\|_X \le (\sup_{t\in i\mathbb{R}} \|f(it)\|_X)^{1-\theta} (\sup_{t\in i\mathbb{R}} \|f(1+it)\|_X)^\theta,\ \ 0 < \theta < 1.$$

This estimate implies that if f vanishes in $i\mathbb{R}$ or in $1 + i\mathbb{R}$ then f vanishes in S.
(Hint: apply the maximum principle of exercise 1 to $\varphi(z) = e^{\lambda z} f(z)$ and then choose $\lambda > 0$ properly).

3) Show that $[X, X]_\theta = X$, with identical norms.

4) Using Lemma 2.1.2 prove that $X \cap Y$ is dense in $[X, Y]_\theta$ for every $\theta \in (0, 1)$.

Chapter 3

Interpolation and domains of operators

3.1 Operators with rays of minimal growth

Let X be a real or complex Banach space with norm $\|\cdot\|$. In this section we consider a linear operator $A : D(A) \subset X \mapsto X$ such that

$$\rho(A) \supset (0,\infty), \ \ \exists M: \ \|\lambda R(\lambda, A)\|_{L(X)} \le M, \ \ \lambda > 0. \tag{3.1}$$

Since $\rho(A)$ is not empty, then A is a closed operator, so that $D(A)$ is a Banach space with the graph norm $\|x\|_{D(A)} = \|x\| + \|Ax\|$. Moreover for every $m \in \mathbb{N}$ also A^m is a closed operator (see exercise 1, §3.2.1).

This section is devoted to the study of the real interpolation spaces $(X, D(A))_{\theta,p}$, and, more generally, $(X, D(A^m))_{\theta,p}$. Since for every t, $\omega \in \mathbb{R}$ the graph norm of $D(A^m)$ is equivalent to the graph norm of $D(B^m)$ with $B = e^{it}(A+\omega I)$, the case of an operator B satisfying

$$\rho(B) \supset \{\lambda e^{i\theta} : \ \lambda > \lambda_0\}, \ \ \exists M: \ \|\lambda R(\lambda e^{i\theta}, B)\|_{L(X)} \le M, \ \ \lambda > \lambda_0$$

for some $\theta \in [0, 2\pi)$, $\lambda_0 \ge 0$, may be easily reduced to this one. The halfline $r = \{\lambda e^{i\theta} : \lambda > \lambda_0\}$ is said to be a *ray of minimal growth* of the resolvent of B. See [2, Def. 2.1].

Proposition 3.1.1 *Let A satisfy (3.1). Then*

$$(X, D(A))_{\theta,p} = \{x \in X : \ \lambda \mapsto \varphi(\lambda) = \lambda^{\theta}\|AR(\lambda, A)x\| \in L^p_*(0,+\infty)\}$$

and the norms $\|x\|_{\theta,p}$ and

$$\|x\|^*_{\theta,p} = \|x\| + \|\varphi\|_{L^p_*(0,+\infty)}$$

are equivalent.

Proof. Let $x \in (X, D(A))_{\theta,p}$. Then if $x = a + b$ with $a \in X$, $b \in D(A)$, for every $\lambda > 0$ we have

$$\begin{aligned} \lambda^{\theta}\|AR(\lambda, A)x\| &\le \lambda^{\theta}\|AR(\lambda, A)a\| + \lambda^{\theta}\|R(\lambda, A)Ab\| \\ &\le (M+1)\lambda^{\theta}\|a\| + M\lambda^{\theta-1}\|Ab\| \\ &\le (M+1)\lambda^{\theta}(\|a\| + \lambda^{-1}\|b\|_{D(A)}), \end{aligned}$$

so that

$$\lambda^\theta \|AR(\lambda, A)x\| \le (M+1)\lambda^\theta K(\lambda^{-1}, x).$$

With the changement of variables $\lambda \mapsto \lambda^{-1}$ we see that the right hand side belongs to $L^p_*(0,\infty)$, with norm equal to $(M+1)\|x\|_{\theta,p}$. Therefore $\varphi(\lambda) = \lambda^\theta\|AR(\lambda, A)x\|$ is in $L^p_*(0,\infty)$, and

$$\|x\|^*_{\theta,p} \le (M+1)\|x\|_{\theta,p}.$$

Conversely, if $\varphi \in L^p_*(0,\infty)$, set for every $\lambda \ge 1$

$$x = a_\lambda + b_\lambda = -AR(\lambda, A)x + \lambda R(\lambda, A)x,$$

so that

$$\lambda^\theta K(\lambda^{-1}, x) \le \lambda^\theta(\|AR(\lambda, A)x\| + \lambda^{-1}\|\lambda R(\lambda, A)x\|_{D(A)})$$

$$= \lambda^\theta(2\|AR(\lambda, A)x\| + \|R(\lambda, A)x\|).$$

The right hand side belongs to $L^p_*(1,\infty)$, with norm estimated by

$$2\|x\|^*_{\theta,p} + M\left(\frac{1}{(1-\theta)p}\right)^{1/p}\|x\|, \;\; p<\infty,$$

$$2\|x\|^*_{\theta,\infty} + M\|x\|, \;\; p=\infty.$$

It follows that $t \mapsto t^{-\theta}K(t,x) \in L^p_*(0,1)$, and hence $x \in (X, D(A))_{\theta,p}$ and

$$\|x\|_{\theta,p} \le C_p(\|x\|^*_{\theta,p} + \|x\|).$$

□

The following notation is widely used.

Definition 3.1.2 *For $0<\theta<1$, $1 \le p \le \infty$ we set*

$$D_A(\theta, p) = (X, D(A))_{\theta,p}.$$

For $0<\theta<1$, $k \in \mathbb{N}$, $1 \le p \le \infty$ we set

$$D_A(\theta+k, p) = \{x \in D(A^k) : \; A^k x \in D_A(\theta, p)\},$$

$$\|x\|_{D_A(\theta+k,p)} = \|x\| + \|A^k x\|_{D_A(\theta,p)},$$

that is, $D_A(\theta+k,p)$ is the domain of the part of A^k in $D_A(\theta,p)$.

From the definition we get easily

Lemma 3.1.3 *For $0<\theta<1$, $1 \le p \le \infty$,*

$$D_A(\theta+1, p) = (D(A), D(A^2))_{\theta,p}$$

and, more generally,

$$D_A(\theta+k, p) = (D(A^k), D(A^{k+1}))_{\theta,p}.$$

Proof. It is sufficient to remark that $(I-A)^k$ is an isomorphism from $D(A^k)$ to X, and also from $D(A^{k+1})$ to $D(A)$. By the interpolation theorem 1.1.6, it is an isomorphism between $(D(A^k), D(A^{k+1}))_{\theta,p}$ and $(X, D(A))_{\theta,p}$, and the statement follows. □

It is also important to characterize the real interpolation spaces between X and $D(A^2)$, or more generally, between X and $D(A^m)$. The following proposition is useful.

Proposition 3.1.4 *Let A satisfy (3.1). Then $D(A) \in J_{1/2}(X, D(A^2)) \cap K_{1/2}(X, D(A^2))$.*

Proof. Let us prove that $D(A) \in J_{1/2}(X, D(A^2))$. For every $x \in D(A)$ it holds

$$\lim_{\lambda\to\infty} \lambda R(\lambda, A)x = \lim_{\lambda\to\infty} R(\lambda, A)Ax + x = x.$$

Setting $f(\sigma) = \sigma R(\sigma, A)x$ for $\sigma > 0$, we have

$$f'(\sigma) = R(\sigma, A)x - \sigma R(\sigma, A)^2 x = R(\sigma, A)(I - \sigma R(\sigma, A))x = -R(\sigma, A)^2 Ax$$

and $f(+\infty) = x$, so that

$$x - \lambda R(\lambda, A)x = -\int_\lambda^\infty R(\sigma, A)^2 Ax d\sigma, \quad \lambda > 0,$$

and if $x \in D(A^2)$,

$$Ax = \lambda A R(\lambda, A)x - \int_\lambda^\infty R(\sigma, A)^2 A^2 x d\sigma, \quad \lambda > 0.$$

Therefore,

$$\|Ax\| \le \lambda(M+1)\|x\| + \frac{M^2}{\lambda}\|A^2 x\|, \quad \lambda > 0.$$

Taking the infimum for $\lambda \in (0, \infty)$ we get

$$\|Ax\| \le 2M(M+1)^{1/2}\|x\|^{1/2}\|A^2 x\|^{1/2}, \quad x \in D(A^2), \tag{3.2}$$

so that

$$\|x\|_{D(A)} \le C\|x\|^{1/2}\|x\|^{1/2}_{D(A^2)}, \quad x \in D(A^2),$$

that is, $D(A) \in J_{1/2}(X, D(A^2))$.

Let us prove that $D(A) \in K_{1/2}(X, D(A^2))$. For every $x \in D(A)$ split x as

$$x = -R(\lambda, A)Ax + \lambda R(\lambda, A)x, \quad \lambda > 0,$$

where

$$\|R(\lambda, A)Ax\| \le \frac{M}{\lambda}\|x\|_{D(A)},$$

$$\|\lambda R(\lambda, A)x\|_{D(A^2)} = \|\lambda R(\lambda, A)x\| + \|\lambda A R(\lambda, A)Ax\|$$

$$\le M\|x\| + \lambda(M+1)\|Ax\|$$

so that setting $t = \lambda^{-2}$

$$K(t, x, X, D(A^2)) \le \|R(t^{-1/2}, A)Ax\| + t\|t^{-1/2}R(t^{-1/2}, A)x\|_{D(A^2)}$$

$$\le Mt^{1/2}\|x\|_{D(A)} + Mt\|x\| + (M+1)t^{1/2}\|Ax\|, \quad t > 0$$

which implies that $t \mapsto K(t,x,X,D(A^2))$ is bounded in $(0,1]$ by $(2M+1)\|x\|_{D(A)}$. Since it is bounded by $\|x\|$ in $(1,\infty)$, then $x \in (X,D(A^2))_{1/2,\infty}$ and

$$\|x\|_{(X,D(A^2))_{1/2,\infty}} \leq (2M+1)\|x\|_{D(A)}.$$

□

But in general $D(A)$ is not an interpolation space between X and $D(A^2)$. As a counterexample we may take $X = C_b(\mathbb{R})$, $A =$ realization of $\partial/\partial x$ in X. See example 1.3.3.

As a corollary of proposition 3.1.4 we get a useful characterization of $(X,D(A^2))_{\theta,p}$.

Proposition 3.1.5 *Let A satisfy (3.1). Then for $\theta \neq 1/2$*

$$(X,D(A^2))_{\theta,p} = D_A(2\theta,p).$$

Proof. Taking into account that X belongs to $J_0(X,D(A^2)) \cap K_0(X,D(A^2))$ and $D(A)$ belongs to $J_{1/2}(X,D(A^2)) \cap K_{1/2}(X,D(A^2))$, and applying the Reiteration Theorem with $E_0 = X$, $E_1 = D(A)$ we get

$$D_A(\alpha,p) = (X,D(A))_{\alpha,p} = (X,D(A^2))_{\alpha/2,p}, \;\; 0<\alpha<1,$$

and setting $\alpha = 2\theta$ the statement follows for $0<\theta<1/2$. Taking into account that $D(A^2)$ belongs to $J_1(X,D(A^2)) \cap K_1(X,D(A^2))$ and $D(A)$ belongs to $J_{1/2}(X,D(A^2)) \cap K_{1/2}(X,D(A^2))$, and applying the Reiteration Theorem with $E_0 = D(A)$, $E_1 = D(A^2)$ we get

$$D_A(\alpha+1,p) = (D(A),D(A^2))_{\alpha,p} = (X,D(A^2))_{(\alpha+1)/2,p}, \;\; 0<\alpha<1,$$

and setting $\alpha+1 = 2\theta$ the statement follows for $1/2<\theta<1$. □

Another characterization, which holds also for $\theta = 1/2$, is the following one.

Proposition 3.1.6 *Let A satisfy (3.1). Then for $0<\theta<1$, $1 \leq p \leq \infty$*

$$(X,D(A^2))_{\theta,p} = \{x \in X : \; \lambda \mapsto \widetilde{\varphi}(\lambda) = \lambda^{2\theta}\|(AR(\lambda,A))^2x\| \in L^p_*(0,\infty)\},$$

and the norms $\|x\|_{(X,D(A^2))_{\theta,p}}$ and

$$\|x\|\widetilde{}_{\theta,p} = \|x\| + \|\widetilde{\varphi}\|_{L^p_*(0,\infty)}$$

are equivalent.

Proof. The proof is very close to the proof of proposition 3.1.1. Let $x \in (X,D(A^2))_{\theta,p}$. Then if $x = a+b$ with $a \in X$, $b \in D(A^2)$, for every $\lambda>0$ we have

$$\lambda^{2\theta}\|(AR(\lambda,A))^2x\| \leq \lambda^{2\theta}\|(AR(\lambda,A))^2a\| + \lambda^{2\theta}\|R(\lambda,A)^2A^2b\|$$

$$\leq (M+1)^2\lambda^{2\theta}\|a\| + M^2\lambda^{2\theta-2}\|A^2b\| \leq (M+1)^2\lambda^{2\theta}(\|a\| + \lambda^{-2}\|b\|_{D(A^2)}),$$

so that

$$\lambda^{2\theta}\|(AR(\lambda,A))^2x\| \leq (M+1)^2\lambda^{2\theta}K(\lambda^{-2},x).$$

We know that $\lambda^{-\theta}K(\lambda,x)\in L^p_*(0,\infty)$. With the change of variable $\xi=\lambda^{-2}$ we get that $\lambda\mapsto\lambda^{2\theta}K(\lambda^{-2},x)\in L^p_*(0,\infty)$, with norm equal to $2^{-1/p}\|x\|_{\theta,p}$. Therefore $\widetilde{\varphi}(\lambda)=\lambda^{2\theta}\|(AR(\lambda,A))^2x\|$ is in $L^p_*(0,\infty)$, and

$$\|x\|^{\widetilde{}}_{\theta,p}\leq 2^{-1/p}(M+1)^2\|x\|_{\theta,p}.$$

(The formula is true also for $p=\infty$ if we set $1/\infty=0$).

Conversely, if $\varphi\in L^p_*(0,\infty)$, from the obvious identity

$$x=\lambda^2R(\lambda,A)^2x-2\lambda AR(\lambda,A)^2x+A^2R(\lambda,A)^2x,$$

where

$$\lambda AR(\lambda,A)^2x=\lambda(\lambda-A)AR(\lambda,A)^3x$$

$$=AR(\lambda,A)\lambda^2R(\lambda,A)^2x-\lambda R(\lambda,A)A^2R(\lambda,A)^2x$$

we get

$$x=(I-2AR(\lambda,A))\lambda^2R(\lambda,A)^2x+(2\lambda R(\lambda,A)+I)A^2R(\lambda,A)^2x,\lambda\geq 1,$$

where

$$\|(I-2AR(\lambda,A))\lambda^2R(\lambda,A)^2x\|_{D(A^2)}$$

$$=\|(I-2AR(\lambda,A))\lambda^2R(\lambda,A)^2x\|+\|(I-2AR(\lambda,A))\lambda^2A^2R(\lambda,A)^2x\|$$

$$\leq(2M+3)M^2\|x\|+(2M+3)\lambda^2\|A^2R(\lambda,A)^2x\|$$

and

$$\|(2\lambda R(\lambda,A)+I)A^2R(\lambda,A)^2x\|\leq(2M+1)\|A^2R(\lambda,A)^2x\|.$$

Therefore,

$$\lambda^{2\theta}K(\lambda^{-2},x,X,D(A^2))$$

$$\leq\lambda^{2\theta}(\|(2\lambda R(\lambda,A)+I)A^2R(\lambda,A)^2x\|$$

$$+\lambda^{-2}\|(I-2AR(\lambda,A))\lambda^2R(\lambda,A)^2x\|_{D(A^2)})$$

$$\leq(4M+4)\lambda^{2\theta}\|A^2R(\lambda,A)^2x\|+(2M+3)M^2\lambda^{2\theta-2}\|x\|.$$

The right hand side belongs to $L^p_*(1,\infty)$, with norm estimated by

$$(4M+3)\|x\|^{\widetilde{}}_{\theta,p}+(2M+2)M^2\left(\frac{1}{(2-2\theta)p}\right)^{1/p}\|x\|,$$

which is true also for $p=\infty$ with the convention $(1/\infty)^{1/\infty}=1$. It follows that $t\mapsto t^{-\theta}K(t,x,X,D(A^2))\in L^p_*(0,1)$, and hence $x\in(X,D(A^2))_{\theta,p}$ and

$$\|x\|_{(X,D(A^2))_{\theta,p}}\leq C_p(\|x\|^{\widetilde{}}_{\theta,p}+\|x\|).$$

□

Propositions 3.1.4 and 3.1.5 may be generalized as follows.

Proposition 3.1.7 *Let A satisfy (3.1), and let r, $m \in \mathbb{N}$, $r > m$. Then $D(A^r) \in J_{r/m}(X, D(A^m)) \cap K_{r/m}(X, D(A^m))$.*

Proposition 3.1.8 *Let A satisfy (3.1), and let $m \in \mathbb{N}$. Then for $\theta \in (0,1)$ such that $\theta m \notin \mathbb{N}$, and for $1 \le p \le \infty$*

$$(X, D(A^m))_{\theta,p} = D_A(m\theta, p).$$

3.1.1 Two or more operators

Let us consider now two operators $A : D(A) \mapsto X$, $B : D(B) \mapsto X$, both satisfying (3.1). Throughout the section we shall assume that A and B commute, in the sense that

$$R(\lambda, A)R(\lambda, B) = R(\lambda, B)R(\lambda, A), \quad \lambda > 0.$$

It follows that $D(A^k B^h) = D(B^h A^k)$ for all natural numbers h, k, and that $A^k B^h x = B^h A^k x$ for every x in $D(A^k B^h)$.

Definition 3.1.9 *For every $m \in \mathbb{N}$ set*

$$K^m = \bigcap_{j=0}^{m} D(A^j B^{m-j}), \quad \|x\|_{K^m} = \|x\| + \sum_{j=0}^{m} \|A^j B^{m-j} x\|.$$

The main result of the section is the following.

Theorem 3.1.10 *Let $m \in \mathbb{N}$, $p \in [1, \infty]$ and $\theta \in (0,1)$ be such that $m\theta$ is not integer, and set $k = [m\theta]$, $\sigma = \{m\theta\}$. Then we have*

$$(X, K^m)_{\theta,p} = \{x \in K^k : A^j B^{k-j} x \in D_A(\sigma, p) \cap D_B(\sigma, p), \;\; j = 0, \dots, k\},$$

and the norms

$$x \mapsto \|x\|_{(X,K^m)_{\theta,p}},$$

$$x \mapsto \|x\| + \textstyle\sum_{j=0}^{k}(\|A^j B^{k-j} x\|_{D_A(\sigma,p)} + \|A^j B^{k-j} x\|_{D_A(\sigma,p)})$$

are equivalent.

The theorem will be proved in several steps. The first one is the case $m = 1$.

Proposition 3.1.11 *For every $p \in [1, \infty]$ and $\theta \in (0,1)$ we have*

$$(X, K^1)_{\theta,p} = D_A(\theta, p) \cap D_B(\theta, p),$$

and the norms

$$\|x\|_{(X,K^1)_{\theta,p}}, \quad \|x\|_{D_A(\theta,p)} + \|x\|_{D_B(\theta,p)}$$

are equivalent.

Proof. The embedding $(X,K^1)_{\theta,p} \subset D_A(\theta,p)\cap D_B(\theta,p)$ is obvious, since $K^1 = D(A)\cap D(B)$ is continuously embedded both in $D(A)$ and in $D(B)$.

Let $x \in D_A(\theta,p)\cap D_B(\theta,p)$. We recall (see proposition 3.1.1) that the functions

$$\lambda \mapsto \lambda^\theta \|AR(\lambda,A)x\|,\ \ \lambda \mapsto \lambda^\theta \|BR(\lambda,B)x\|,\ \ \lambda>0,$$

belong to $L^p_*(0,\infty)$ and their norms are less than $C\|x\|_{D_A(\theta,p)}$, $C\|x\|_{D_B(\theta,p)}$, respectively.

For every $\lambda>0$ set

$$v(\lambda) = \lambda^2 R(\lambda,A)R(\lambda,B)x,\ \ \lambda>0, \tag{3.3}$$

and split $x = x - v(\lambda) + v(\lambda)$. It holds

$$\|v(\lambda)-x\| \le \|\lambda R(\lambda,A)(\lambda R(\lambda,B)x - x)\| + \|\lambda R(\lambda,A)x - x\|$$

$$\le M\|BR(\lambda,B)x\| + \|AR(\lambda,A)x\|,$$

and

$$\|v(\lambda)\|_{K^1} = \|v(\lambda)\| + \|Av(\lambda)\| + \|Bv(\lambda)\|$$

$$\le M^2\|x\| + \lambda M\|AR(\lambda,A)x\| + \lambda M\|BR(\lambda,B)x\|.$$

Therefore, for $\lambda \ge 1$

$$\lambda^\theta K(\lambda^{-1},x,X,K^1) \le 2M\lambda^\theta(\|AR(\lambda,A)x\| + \|BR(\lambda,B)x\|) + M^2\lambda^{\theta-1}\|x\|,$$

so that $\lambda \mapsto \lambda^\theta K(\lambda^{-1},x,X,K^1) \in L^p_*(1,\infty)$, with norm estimated by const. $(\|x\| + \|x\|_{D_A(\theta,p)} + \|x\|_{D_B(\theta,p)})$. Then $\lambda \mapsto \lambda^{-\theta}K(\lambda,x,X,K^1) \in L^p_*(0,1)$, with the same norm, and the statement follows. □

As a second step we show that

Proposition 3.1.12 *For every $p\in[1,\infty]$ and $\theta\in(0,1)$ we have*

$$(K^1,K^2)_{\theta,p} = \{x\in K^1 :\ Ax,\ Bx \in D_A(\theta,p)\cap D_B(\theta,p)\}$$

$$= D_A(\theta+1,p)\cap D_B(\theta+1,p),$$

and the norms

$$x \mapsto \|x\|_{(K^1,K^2)_{\theta,p}},$$

$$x \mapsto \|x\| + \|Ax\|_{D_A(\theta,p)} + \|Ax\|_{D_B(\theta,p)} + \|Bx\|_{D_A(\theta,p)} + \|Bx\|_{D_B(\theta,p)},$$

$$x \mapsto \|x\|_{D_A(\theta+1,p)} + \|x\|_{D_B(\theta+1,p)}$$

are equivalent.

Proof. Let us prove the embeddings $\subset$. Since $K^1 \subset D(A)$ and $K^2 \subset D(A^2)$ then $(K^1,K^2)_{\theta,p} \subset (D(A),D(A^2))_{\theta,p} = D_A(\theta+1,p)$. Similarly, $(K^1,K^2)_{\theta,p} \subset D_B(\theta+1,p)$. It remains to show that each $x\in(K^1,K^2)_{\theta,p}$ is such that $Ax\in D_B(\theta,p)$ and $Bx\in D_A(\theta,p)$. For every $a\in K^1$, $b\in K^2$ such that $x=a+b$ we have

$$\lambda^\theta\|BR(\lambda,B)Ax\| \le \lambda^\theta\|BR(\lambda,B)Aa\| + \lambda^\theta\|BR(\lambda,B)Ab\|$$

$$\le \lambda^\theta(M+1)\|Aa\| + M\lambda^{\theta-1}\|BAb\| \le (M+1)\lambda^\theta(\|a\|_{K^1} + \lambda^{-1}\|b\|_{K^2})$$

so that

$$\lambda^\theta \|BR(\lambda, B)Ax\| \le (M+1)\lambda^\theta K(\lambda^{-1}, x, K^1, K^2), \;\; \lambda > 0.$$

It follows that $\lambda^\theta \|BR(\lambda, B)Ax\| \in L^p_*(0,\infty)$ with norm not exceeding $(M+1)\|x\|_{(K^1,K^2)_{\theta,p}}$, and the embedding $\subset$ is proved.

The proof of the embedding $\{x \in K^1 :\, Ax,\, Bx \in D_A(\theta,p) \cap D_B(\theta,p)\} \subset (K^1,K^2)_{\theta,p}$ is similar to the corresponding proof in proposition 3.1.11, and is omitted.

Let us prove that $D_A(\theta+1,p) \cap D_B(\theta+1,p) \subset \{x \in K^1 :\, Ax,\, Bx \in D_A(\theta,p) \cap D_B(\theta,p)\}$. We have only to show that if $x \in D_A(\theta+1,p) \cap D_B(\theta+1,p)$ then $Ax \in D_B(\theta,p)$ and $Bx \in D_A(\theta,p)$. Indeed, for each $\lambda > 0$ we have

$$\begin{aligned}
&\lambda^\theta \|B^2R(\lambda,B)^2Ax\| \\
&\le \lambda^{\theta+1}\|B^2R(\lambda,B)^2R(\lambda,A)Ax\| + \lambda^\theta\|B^2R(\lambda,B)^2AR(\lambda,A)Ax\| \\
&\le \lambda^\theta M(M+1)\|BR(\lambda,B)Bx\| + \lambda^\theta(M+1)^2\|AR(\lambda,A)Ax\|
\end{aligned}$$

so that $\lambda \mapsto \lambda^\theta\|B^2R(\lambda,B)^2Ax\| \in L^p_*(0,\infty)$ with norm not exceeding

$$M(M+1)\|Bx\|^*_{D_B(\theta,p)} + (M+1)^2\|Ax\|^*_{D_A(\theta,p)}.$$

Thanks to proposition 3.1.6, $Ax \in D_{B^2}(\theta/2,p)$, which coincides with $D_B(\theta,p)$ thanks to proposition 3.1.5. So, $Ax \in D_B(\theta,p)$ and $\|Ax\|_{D_B(\theta,p)} \le C(\|x\|_{D_A(\theta+1,p)} + \|x\|_{D_B(\theta+1,p)}$. Similarly, $Bx \in D_A(\theta,p)$ and $\|Bx\|_{D_A(\theta,p)} \le C(\|x\|_{D_B(\theta+1,p)} + \|x\|_{D_A(\theta+1,p)})$. □

In the last part of the proof of proposition 3.1.12 we have shown that if $x \in D_A(\theta+1,p) \cap D_B(\theta+1,p)$ then $Ax \in D_B(\theta,p)$ and $Bx \in D_A(\theta,p)$, a sort of "mixed regularity" result. However it is not true in general that $D(A^2) \cap D(B^2) \subset D(AB)$.

For instance, let A be the realization of $\partial/\partial x$ and let B be the realization of $\partial/\partial y$ in $X = C(\mathbb{R}^2)$. Then $D_A(\theta+1,\infty)$ consists of the functions $f \in X$ such that $x \mapsto f(x,y) \in C^{\theta+1}(\mathbb{R})$, uniformly with respect to $y \in \mathbb{R}$, and similarly $D_B(\theta+1,\infty)$ consists of the functions $f \in X$ such that $y \mapsto f(x,y) \in C^{\theta+1}(\mathbb{R})$, uniformly with respect to $x \in \mathbb{R}$. Proposition 3.1.12 states that if $f \in D_A(\theta+1,\infty) \cap D_B(\theta+1,\infty)$ then $\partial f/\partial x \in D_B(\theta,\infty)$, that is it is Hölder continuous also with respect to y, and $\partial f/\partial y \in D_A(\theta,\infty)$, that is it is Hölder continuous also with respect to x. On the other hand, it is known that in this example $D(A^2) \cap D(B^2)$ is not embedded in $D(AB)$.

A similar proof yields

Proposition 3.1.13 *For every $k \in \mathbb{N}$, $p \in [1,\infty]$ and $\theta \in (0,1)$ we have*

$$(K^k, K^{k+1})_{\theta,p} = \{x \in K^k :\, A^jB^{k-j}x \in D_A(\theta,p) \cap D_B(\theta,p),\; j = 0,\dots,k\}$$

and the norms

$$\|x\|_{(K^k,K^{k+1})_{\theta,p}}, \;\; \|x\| + \textstyle\sum_{j=0}^k (\|A^jB^{k-j}x\|_{D_A(\theta,p)} + \|A^jB^{k-j}x\|_{D_B(\theta,p)})$$

are equivalent.

Next step consists in proving that K^1 belongs to $J_{1/2}(X,K^2)$ and to $K_{1/2}(X,K^2)$.

Proposition 3.1.14

$$K^1 \in J_{1/2}(X,K^2) \cap K_{1/2}(X,K^2).$$

Proof. We already know that $D(A) \in J_{1/2}(X, D(A^2))$ and that $D(B) \in J_{1/2}(X, D(B^2))$. Therefore there is $C > 0$ such that

$$\|x\|_{K^1} \le \|x\|_{D(A)} + \|x\|_{D(B)}$$

$$\le C\|x\|^{1/2}(\|x\|^{1/2}_{D(A^2)} + \|x\|^{1/2}_{D(B^2)}) \le C'\|x\|^{1/2}\|x\|^{1/2}_{K^2},$$

which means that $K^1 \in J_{1/2}(X, K^2)$.

To prove that $K^1 \in K_{1/2}(X, K^2)$, for every $x \in K^1$ we split again $x = x - v(\lambda) + v(\lambda)$ for every $\lambda > 0$, where v is the function defined in (3.3). Then

$$\|v(\lambda) - x\| \le \|\lambda R(\lambda, A)(\lambda R(\lambda, B)x - x)\| + \|\lambda R(\lambda, A)x - x\|$$

$$= \|\lambda R(\lambda, A)R(\lambda, B)Bx\| + \|R(\lambda, A)Ax\|$$

$$\le \lambda^{-1}(M(M+1)\|Bx\| + M\|Ax\|),$$

and

$$\|v(\lambda)\|_{K^2} = \|v(\lambda)\| + \|A^2v(\lambda)\| + \|ABv(\lambda)\| + \|B^2v(\lambda)\|$$

$$\le M^2\|x\| + 2M(M+1)\lambda(\|Ax\| + \|Bx\|).$$

Setting $\lambda = t^{-1/2}$ we deduce that

$$t^{-1/2}K(t, x, X, K^2) \le t^{-1/2}(\|x - v(t^{-1/2})\| + t\|v(t^{-1/2})\|_{K^2})$$

$$\le C(\|x\|_{K^1} + t^{1/2}\|x\|),$$

is bounded in $(0, 1)$. We know already that $t \mapsto t^{-1/2}K(t, x, X, K^2)$ is bounded in $[1, \infty)$. Therefore K^1 is in the class $K_{1/2}$ between X and K^2. □

Arguing similarly one shows that

Proposition 3.1.15 *For every $k \in \mathbb{N} \cup \{0\}$, $K^{k+1} \in J_{1/2}(K^k, K^{k+2}) \cap K_{1/2}(K^k, K^{k+2})$. More generally, $K^{k+1} \in J_{1/s}(K^k, K^{k+s}) \cap K_{1/s}(K^k, K^{k+s})$.*

The Reiteration Theorem and proposition 3.1.14 yield now

Proposition 3.1.16 *Let $p \in [1, \infty]$, $\theta \in (0, 1)$, $\theta \ne 1/2$. Then*

$$(X, K^2)_{\theta,p} = D_A(2\theta, p) \cap D_B(2\theta, p),$$

and for $\theta > 1/2$ we have also

$$(X, K^2)_{\theta,p} = \{x \in K^1 : Ax,\ Bx \in D_A(2\theta - 1, p) \cap D_B(2\theta - 1, p)\},$$

with equivalence of the respective norms.

Proof. For $\theta < 1/2$ we apply the Reiteration Theorem with $Y = K^2$, $E_0 = X$, $E_1 = K^1$, and the statement follows from proposition 3.1.11. For $\theta > 1/2$ we apply the Reiteration Theorem with $Y = K^2$, $E_0 = K^1$, $E_1 = K^2$, and the statement follows from proposition 3.1.12. □

The above proposition is a special case of theorem 3.1.10, with $m = 2$. Theorem 3.1.10 in its full generality may be proved by recurrence, arguing similarly. See the exercises of §3.1.2.

The results and the procedures of this section are easily extended to the case of a finite number of operators.

3.1.2 Exercises

1) Let $A : D(A) \subset X \mapsto X$ satisfy (3.1). Prove that for every $m \in \mathbb{N}$, A^m is a closed operator. (Hint: use estimate (3.2)).

2) Prove proposition 3.1.7. Hint: to show that $D(A^r) \in K_{r/m}(X, D(A^m))$ prove preliminarly that $D(A^r) \in K_{1/(m-r)}(D(A^{r-1}), D(A^m))$, using a procedure similar to the one of proposition 3.1.4, and then argue by reiteration.

3) Prove proposition 3.1.8.

4) Prove proposition 3.1.13.

5) Prove proposition 3.1.15. Hint: for the first statement, follow step by step the proof of 3.1.14; for the second statement replace $v(\lambda)$ by $w(\lambda) = \lambda^{2s}R(\lambda, A)^s R(\lambda, B)^s x$.

6) Prove theorem 3.1.10 by recurrence on m, using the procedure of proposition 3.1.16 and the results of propositions 3.1.13 and 3.1.15.

7) Prove that $(0, +\infty)$ is a ray of minimal growth for the following operators:

(a) $A : D(A) = C^1_b(\mathbb{R}) \mapsto C_b(\mathbb{R})$ (resp. $A : D(A) = W^{1,p}(\mathbb{R}) \mapsto L^p(\mathbb{R})$, $1 \le p < \infty$), $Af = f'$

(b) $A : D(A) = C^2_b(\mathbb{R}) \mapsto C_b(\mathbb{R})$ (resp. $A : D(A) = W^{2,p}(\mathbb{R}) \mapsto L^p(\mathbb{R})$, $1 \le p < \infty$), $Af = f''$

(c) $A : D(A) = \{f \in C^2([0,\pi]) :\ f(0) = f(\pi) = 0\} \mapsto C([0,\pi])$ (resp. $A : D(A) = W^{2,p}(0,\pi) \cap W^{1,p}_0(0,\pi) \mapsto L^p(0,\pi)$, $1 \le p\infty$), $Af = f''$

3.2 The case where A generates a semigroup

Let $A : D(A) \subset X \mapsto X$ satisfy (3.1).

Due to the Hille-Yosida Theorem, if in addition $D(A)$ is dense in X and for every $n \in \mathbb{N}$ $\|(\lambda R(\lambda, A))^n\|_{L(X)} \le M$, then A is the infinitesimal generator of a strongly continuous semigroup $T(t)$, and the following representation formula holds.

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t} T(t)dt, \quad \lambda > 0. \tag{3.4}$$

Since $AR(\lambda, A) = \lambda R(\lambda, A) - I$, then

$$AR(\lambda, A) = \int_0^\infty \lambda e^{-\lambda t}(T(t) - I)dt, \quad \lambda > 0. \tag{3.5}$$

Proposition 3.2.1 *Let A generate a semigroup $T(t)$. Then*

$$(X, D(A))_{\theta,p} = \{x \in X :\ t \mapsto \psi(t) = t^{-\theta}\|T(t)x - x\| \in L^p_*(0,\infty)\}$$

and the norms $\|x\|_{\theta,p}$ and

$$\|x\|^{**}_{\theta,p} = \|x\| + \|\psi\|_{L^p_*(0,\infty)}$$

are equivalent.

Proof. Recall that for every $b \in D(A)$ we have

$$T(t)b - b = \int_0^t AT(s)b\,ds = \int_0^t T(s)Ab\,ds, \quad t > 0.$$

Let $x \in (X, D(A))_{\theta,p}$. Then if $x = a + b$ with $a \in X$, $b \in D(A)$, for every $t > 0$ we have

$$t^{-\theta}\|T(t)x - x\| \le t^{-\theta}(\|T(t)a - a\| + \|T(t)b - b\|)$$

$$\le t^{-\theta}((M+1)\|a\| + tM\|Ab\|) \le (M+1)t^{-\theta}K(t,x).$$

Therefore $\psi(t) = t^{-\theta}\|T(t)x - x\| \in L^p_*(0,\infty)$ and

$$\|x\|^{**}_{\theta,p} \le (M+1)\|x\|_{\theta,p}.$$

Conversely, if $\psi \in L^p_*(0,\infty)$ let us use (3.5) to get

$$\lambda^\theta\|AR(\lambda, A)x\| \le \int_0^\infty \lambda^{\theta+1}t^{\theta+1}e^{-\lambda t}\frac{\|T(t)x - x\|}{t^\theta}\frac{dt}{t},$$

that is, φ is the multiplicative convolution between the functions $f(t) = t^{\theta+1}e^{-t}$ and $\psi(t) = t^{-\theta}\|T(t)x - x\|$. Since $f \in L^1_*(0,\infty)$ and $\psi \in L^p_*(0,\infty)$, then $\varphi \in L^p_*(0,\infty)$ and $\|\varphi\|_{L^p_*(0,\infty)} \le \|f\|_{L^1_*(0,\infty)}\|\psi\|_{L^p_*(0,\infty)}$, so that

$$\|x\|^*_{\theta,p} \le \Gamma(\theta+1)\|x\|^{**}_{\theta,p},$$

and the statement follows. □

Proposition 3.2.2 *Under the assumptions of proposition 3.2.1, for every $\theta \in (0,1)$ and $p \in [1,\infty]$ we have*

$$(X, D(A^2))_{\theta,p} = \{x \in X:\ t \mapsto \widetilde{\psi}(t) = t^{-2\theta}\|(T(t) - I)^2x\| \in L^p_*(0,\infty)\}$$

and the norms $\|x\|_{\theta,p}$ and

$$\|x\|^{\sim}_{\theta,p} = \|x\| + \|\widetilde{\psi}\|_{L^p_*(0,\infty)}$$

are equivalent.

Proof. Recall that for every $b \in D(A^2)$ we have

$$(T(t) - I)^2b = (T(t) - I)\int_0^t T(\sigma)Ab\,d\sigma = \int_0^t\int_0^t T(s+\sigma)A^2b\,ds\,d\sigma, \quad t > 0,$$

so that

$$\|(T(t) - I)^2b\| \le t^2M\|A^2b\|.$$

Let $x \in (X, D(A^2))_{\theta,p}$. Then if $x = a + b$ with $a \in X$, $b \in D(A^2)$, for every $t > 0$ we have

$$t^{-2\theta}\|(T(t) - I)^2x\| \le t^{-2\theta}(\|(T(t) - I)^2a\| + \|(T(t) - I)^2b\|)$$

$$\le t^{-2\theta}((M+1)^2\|a\| + t^2M^2\|A^2b\|)$$

so that

$$t^{-2\theta}\|(T(t) - I)^2x\| \le (M+1)^2t^{-2\theta}K(t^2, x).$$

Therefore $\widetilde{\psi}(t) = t^{-2\theta}\|(T(t)-I)^2 x\| \in L^p_*(0,\infty)$ and

$$\|x\|_{\widetilde{\theta,p}} \leq 2^{-1/p}(M+1)^2\|x\|_{\theta,p}.$$

Conversely, let x be such that $\widetilde{\psi}(t) \in L^p_*(0,\infty)$. Then from (3.5) it follows that

$$(AR(\lambda,A))^2 x = \lambda^2 \int_0^\infty \int_0^\infty e^{-\lambda(t+s)}(T(t+s)-T(t)-T(s)+I)x ds\, dt$$

$$= 2\lambda^2 \int_0^\infty e^{-2\lambda u} du \int_0^{2u} (T(2u)-T(t)-T(2u-t)+I)x dt$$

$$= 2\lambda^2 \int_0^\infty e^{-2\lambda u}(T(2u)-2T(u)+I)x \int_0^{2u} dt\, du$$

$$+2\lambda^2 \int_0^\infty e^{-2\lambda u} du \int_0^{2u} (2T(u)-T(t)-T(2u-t))x dt.$$

The first integral is nothing but

$$4\lambda^2 \int_0^\infty u e^{-2\lambda u}(T(u)-I)^2 x du.$$

To rewrite the second one we note that

$$\int_0^{2u} (2T(u)-T(t)-T(2u-t))dt = \int_0^{2u} (2T(u)-2T(t))dt$$

$$= \left(\int_0^u + \int_u^{2u}\right)(2T(u)-2T(t))dt$$

$$= \int_0^u 2(T(u-t)-I)T(t)dt + \int_u^{2u} 2T(u)(I-T(t-u))dt$$

$$= \int_0^u 2(T(s)-I)T(u-s)ds + \int_0^u 2T(u)(I-T(s))ds$$

$$= 2\int_0^u (T(s)-I)(T(u-s)-T(u))ds = -2\int_0^u (T(s)-I)^2 T(u-s)ds.$$

Therefore,

$$\|\lambda^{2\theta}(AR(\lambda,A))^2 x\| \leq 4|(f \star \widetilde{\psi})(\lambda)| + 2|(f \star \widetilde{\psi}_1)(\lambda)|,$$

where $\star$ stands for the multiplicative convolution and

$$f(u) = u^{2+2\theta}e^{-2u}, \quad \widetilde{\psi}_1(u) = \frac{M}{u^{1+2\theta}} \int_0^u \|(T(s)-I)^2 x\| ds.$$

Let us remark now that the Hardy-Young inequality (A.10)(i) implies that if a function z is such that $t \mapsto t^{-\alpha}z(t) \in L^p_*(0,\infty)$ the same is true for its mean $v(t) = t^{-1}\int_0^t z(s)ds$, with

$$\|t \mapsto t^{-\alpha}v(t)\|_{L^p_*(0,\infty)} \leq \frac{1}{(\alpha+1)}\|t \mapsto t^{-\alpha}z(t)\|_{L^p_*(0,\infty)},$$

and this is easily seen to be true also for $p = \infty$. Therefore, $\widetilde{\psi}_1 \in L^p_*(0,\infty)$ and $\|\widetilde{\psi}_1\|_{L^p_*(0,\infty)} \leq (2\theta+1)^{-1}\|\widetilde{\psi}\|_{L^p_*(0,\infty)}$. It follows that $\lambda \mapsto \varphi(\lambda) = \|\lambda^{2\theta}(AR(\lambda,A))^2 x\| \in L^p_*(0,\infty)$, and

$$\|\varphi\|_{L^p_*(0,\infty)} = \|x\|^*_{\theta,p} \leq 4\|f\|_{L^1_*(0,\infty)}(\|\widetilde{\psi}\|_{L^p_*(0,\infty)} + \|\widetilde{\psi}_1\|_{L^p_*(0,\infty)}) \leq C_p\|x\|_{\theta,p},$$

and the statement follows. □

Remark 3.2.3 In the proof of proposition 3.2.1 we have not used the fact that $T(t)$ is strongly continuous or that the domain of A is dense. The only essential assumption is that $T(t)$ is a semigroup such that $\|T(t)\|_{L(X)} \leq M$ and for $\lambda > 0$ the operators

$$R(\lambda) = \int_0^\infty e^{-\lambda t}T(t)dt$$

are well defined and invertible. Indeed, in that case due to the semigroup property $R(\lambda)$ satisfies the resolvent identity $R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu)$, for $\lambda,\ \mu > 0$. From the general spectral theory it follows that there exists a unique closed operator A such that $\rho(A) \supset (0,\infty)$ and $R(\lambda) = R(\lambda, A)$, for every $\lambda > 0$. The results of propositions 3.2.1 and 3.2.2 hold also for such semigroups.

The operator A may still be called generator of $T(t)$, even if it is the infinitesimal generator in the usual sense if and only if $T(t)$ is strongly continuous.

This is the case of the translations semigroups $(T_i(t)f)(x) = f(x+te_i)$ in $X = C_b(\mathbb{R}^n)$, of the Gauss-Weierstrass semigroup

$$P(t)f(x) = \frac{1}{(4\pi t)^{n/2}}\int_{\mathbb{R}^n} e^{-\frac{|y|^2}{4t}}f(x-y)dy,$$

again in $C_b(\mathbb{R}^n)$, of the Ornstein-Uhlenbeck semigroup

$$T(t)f(x) = \frac{1}{(4\pi t)^{n/2}(\det K_t)^{1/2}}\int_{\mathbb{R}^n} e^{-\frac{|K_t^{-1/2}y|^2}{4t}}f(e^{tB}x-y)dy,$$

with $Q \geq 0$, $B \neq 0$ arbitrary $n\times n$ matrices,

$$K_t = \frac{1}{t}\int_0^t e^{sB}Qe^{sB^*}ds,$$

both in $C_b(\mathbb{R}^n)$ and in $BUC(\mathbb{R}^n)$, etc. None of these semigroups is strongly continuous.

A useful embedding result in applications to PDE's is the following.

Theorem 3.2.4 *Let $T(t)$ be a semigroup in X. Assume moreover that there exists a Banach space $E \subset X$ and $m \in \mathbb{N}$, $0 < \beta < 1$, $C > 0$ such that*

$$\|T(t)\|_{L(X,E)} \leq \frac{C}{t^{m\beta}},\ \ t > 0,$$

and that $t \mapsto T(t)x$ is measurable with values in E, for each $x \in X$. Then $E \in J_\beta(X, D(A^m))$, so that $(X, D(A^m))_{\theta\beta,p} \subset (X,E)_{\theta,p}$, for every $\theta \in (0,1)$, $p \in [1,\infty]$.

Proof. Let $x \in D(A^m)$, $\lambda > 0$ and set $(\lambda I - A)^m x = y$. Then $x = (R(\lambda, A))^m y$ so that

$$x = \frac{(-1)^{m-1}}{(m-1)!}\frac{d^{m-1}}{d\lambda^{m-1}}R(\lambda, A)y = \frac{1}{(m-1)!}\int_0^\infty e^{-\lambda s}s^{m-1}T(s)y\,ds,$$

so that for every $\lambda > 0$

$$\|x\|_E \le \frac{C}{(m-1)!}\int_0^\infty e^{-\lambda s}s^{m(1-\beta)-1}ds\|y\| = \frac{C\Gamma(m(1-\beta))}{(m-1)!}\lambda^{m\beta - m}\|y\|$$

$$= \frac{C\Gamma(m(1-\beta))}{(m-1)!}\lambda^{m\beta-m}\left\|\sum_{r=0}^m \binom{m}{r}\lambda^{m-r}(-1)^r A^r x\right\| \le C'\sum_{r=0}^m \lambda^{m\beta - r}\|A^r u\|.$$

Let us recall that $D(A^r)$ belongs to $J_{m/r}(X, D(A^m))$ so that there is C such that $\|x\|_{D(A^r)} \le C\|x\|_{D(A^m)}^{r/m}\|x\|_X^{1-r/m}$. Using such inequalities and then $ab \le C(a^p + b^{p'})$ with $p = n/r$, $p' = r/(n-r)$ we get

$$\|x\|_E \le C\lambda^{m\beta}(\lambda^{-m}\|u\|_{D(A^m)} + \|u\|), \quad \lambda > 0,$$

so that taking the minimum for $\lambda > 0$

$$\|x\|_E \le C\|u\|^{1-\beta}\|u\|_{D(A^m)}^\beta$$

and the statement holds. □

3.2.1 Examples and applications. Schauder type theorems

Example 3.2.5 Let us apply propositions 3.2.1, 3.2.2 to the case $X = L^p(\mathbb{R})$, $1 \le p < \infty$, $A : D(A) = W^{1,p}(\mathbb{R}) \mapsto L^p(\mathbb{R})$, $Af = f'$. Then $T(t)$ is the translations semigroup, $T(t)f(x) = f(x+t)$. Applying proposition 3.1.1 we get for $0 < \theta < 1$

$$(L^p(\mathbb{R}), W^{1,p}(\mathbb{R}))_{\theta,p} = D_A(\theta, p)$$

$$= \{f \in L^p : t \mapsto t^{-\theta}\|f(\cdot + t) - f\|_{L^p_*} \in L^p_*(0,\infty)\} = W^{\theta,p}(\mathbb{R}),$$

which we knew already (example 1.1.8), but this is an alternative proof. Applying proposition 3.1.5 and recalling that $D(A^2) = W^{2,p}(\mathbb{R})$ we get for $\theta \ne 1/2$

$$(L^p(\mathbb{R}), W^{2,p}(\mathbb{R}))_{\theta,p} = D_A(2\theta, p)$$

so that for $\theta < 1/2$

$$(L^p(\mathbb{R}), W^{2,p}(\mathbb{R}))_{\theta,p} = W^{2\theta,p}(\mathbb{R}),$$

and for $\theta > 1/2$, by proposition 3.1.5,

$$(L^p(\mathbb{R}), W^{2,p}(\mathbb{R}))_{\theta,p} = \{f \in W^{1,p} : f' \in D_A(2\theta - 1, p)\} = W^{2\theta,p}(\mathbb{R}).$$

For $\theta = 1/2$ we need proposition 3.1.6: we get

$$(L^p(\mathbb{R}), W^{2,p}(\mathbb{R}))_{1/2,p} =$$

$$= \{f \in L^p : t \mapsto t^{-1}\|f(\cdot + 2t) - 2f(\cdot + t) + f\|_{L^p} \in L^p_*(0,\infty)\} = B^1_{p,p}(\mathbb{R}),$$

which coincides with $W^{1,p}(\mathbb{R})$ only for $p = 2$.

Choosing $X = C_b(\mathbb{R})$, $A : D(A) = C^1_b(\mathbb{R}) \mapsto C_b(\mathbb{R})$, $Af = f'$, we get, recalling remark 3.2.3,

$$(C_b(\mathbb{R}), C^2_b(\mathbb{R}))_{\theta,\infty} = C^{2\theta}_b(\mathbb{R}), \quad \theta \neq 1/2,$$

$$(C_b(\mathbb{R}), C^2_b(\mathbb{R}))_{1/2,\infty} =$$

$$= \left\{ f \in C_b(\mathbb{R}) : \sup_{t\neq 0,\, x\in\mathbb{R}} \frac{|f(x+2t) - 2f(x+t) + f(x)|}{t} < \infty \right\},$$

which Zygmund called $\Lambda^*_1(\mathbb{R})$. It is easy to see that $Lip(\mathbb{R}) \subset \Lambda^*_1(\mathbb{R})$, but the converse is not true.

Example 3.2.6 Let A_i, $i = 1, \ldots, n$ be the realization of the partial derivative $\partial/\partial x_i$ in $C_b(\mathbb{R}^n)$, or in $BUC(\mathbb{R}^n)$, or in $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$. Each A_i satisfies (3.1), with

$$(R(\lambda, A_i)f)(x) = \int_{x_i}^{+\infty} e^{\lambda(x_i - s)} f(x_1, \ldots, x_{i-1}, s, x_{i+1}, \ldots, x_n)ds,$$

for $\lambda > 0$, $f \in X$, $x \in \mathbb{R}^n$, so that $M = 1$ for every i, and $R(\lambda, A_i)R(\lambda, A_j) = R(\lambda, A_j)R(\lambda, A_i)$ for every i, j. We apply theorem 3.1.10 for those θ such that θm is not integer, $\theta m = k + \sigma$, $k = [\theta m]$, $0 < \sigma < 1$.

If $X = C_b(\mathbb{R}^n)$ (resp., $X = BUC(\mathbb{R}^n)$) then $K^m = C^m_b(\mathbb{R}^n)$ (resp., $K^m = BUC^m(\mathbb{R}^n)$). From the second part of proposition 3.1.1, or else from example 1.1.8 we know that $D_{A_i}(\sigma, \infty) = \{f \in X : s \mapsto f(x_1, \ldots, x_{i-1}, s, x_{i+1}, \ldots, x_n) \in C^\sigma_b(\mathbb{R})\}$, so that

$$\bigcap_{i=1}^n D_{A_i}(\sigma, \infty) = C^\sigma_b(\mathbb{R}^n).$$

From theorem 3.1.10 we get

$$(X, K^m)_{\theta,\infty} = \{f \in K^k : D^\alpha f \in C^\sigma(\mathbb{R}^n), |\alpha| = k\} = C^{\theta m}_b(\mathbb{R}^n).$$

Let now $X = L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$. From the second part of proposition 3.1.1 and from example 1.1.8 we know that $D_{A_i}(\sigma, p) = \{f \in X : s \mapsto f(x_1, \ldots, x_{i-1}, s, x_{i+1}, \ldots, x_n) \in W^{\sigma,p}(\mathbb{R})\}$, so that

$$\bigcap_{i=1}^n D_{A_i}(\sigma, p) = W^{\sigma,p}(\mathbb{R}^n).$$

From theorem 3.1.10 we get

$$(X, K^m)_{\theta,p} = (L^p, W^{k,p})_{\theta,p}$$

$$= \{f \in W^{k,p}(\mathbb{R}^n : D^\alpha f \in W^{\sigma,p}(\mathbb{R}^n), |\alpha| = k\} = W^{\theta m,p}(\mathbb{R}^n).$$

After such characterizations we are able to characterize other important interpolation spaces by means of theorem 3.2.4.

Example 3.2.7 *Let A be the realization of the Laplace operator Δ in $X = L^p(\mathbb{R}^n)$, $1 \leq p < \infty$. Then for $0 < \alpha < 1$*

$$W^{\alpha,p}(\mathbb{R}^n) = D_A(\alpha/2, p), \quad W^{\alpha+2,p}(\mathbb{R}^n) = D_A(\alpha/2 + 1, p).$$

If A is the realization of Δ in $X = BUC(\mathbb{R}^n)$, in $X = C_b(\mathbb{R}^n)$ or in $X = L^\infty(\mathbb{R}^n)$, then

$$C^\alpha_b(\mathbb{R}^n) = D_A(\alpha/2, \infty), \quad C^{\alpha+2}_b(\mathbb{R}^n) = D_A(\alpha/2 + 1, \infty).$$

Proof. The embeddings $\subset$ are easy consequences of example 3.2.6. Indeed, let $X = L^p(\mathbb{R}^n)$, $1 \le p < \infty$. Example 3.2.6 yields $W^{\alpha,p}(\mathbb{R}^n) = (L^p(\mathbb{R}^n), W^{2,p}(\mathbb{R}^n))_{\alpha,p}$. Since $W^{2,p} \subset D(A)$, then we have

$$W^{\alpha,p}(\mathbb{R}^n) = (L^p(\mathbb{R}^n), W^{2,p}(\mathbb{R}^n))_{\alpha/2,p} \subset D_A(\alpha/2, p).$$

Similarly, from example 3.2.6 we know that $W^{\alpha+2,p}(\mathbb{R}^n) = (L^p(\mathbb{R}^n), W^{4,p}(\mathbb{R}^n))_{(\alpha+2)/4,p}$. Since $W^{4,p}(\mathbb{R}^n) \subset D(A^2)$, then we have

$$W^{\alpha+2,p}(\mathbb{R}^n) = (L^p(\mathbb{R}^n), W^{4,p}(\mathbb{R}^n))_{(\alpha+2)/4,p}$$

$$\subset (X, D(A^2))_{(\alpha+2)/4,p} = D_A(\alpha/2+1, p),$$

where the last equality follows from proposition 3.1.5.

The same proof works in the case $X = BUC(\mathbb{R}^n)$.

To prove the opposite inclusions we introduce the Gauss-Weierstrass semigroup,

$$P(t)f(x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} f(y)dy, \quad t > 0, \ x \in \mathbb{R}^n. \tag{3.6}$$

$P(t)$ may be seen as a (strongly continuous) semigroup in $X = L^p(\mathbb{R}^n)$, $1 \le p < \infty$ or in $X = BUC(\mathbb{R}^n)$. Its infinitesimal generator is the realization of the Laplace operator in X.

It is easy to see that if $1 \le p \le \infty$, $P(t)f \in C^\infty(\mathbb{R}^n)$ for every $f \in L^p(\mathbb{R}^n)$, and that

$$\|D^\alpha P(t)f\|_{L^p} \le \frac{C_{\alpha,p}}{t^{|\alpha|/2}} \|f\|_{L^p}, \quad t > 0.$$

In particular,

$$\begin{cases} \|P(t)\|_{L(L^p, W^{1,p})} \le C\left(1 + \dfrac{1}{t^{1/2}}\right), & t > 0, \\[2ex] \|P(t)\|_{L(L^p, W^{3,p})} \le C\left(1 + \dfrac{1}{t^{1/2}} + \dfrac{1}{t} + \dfrac{1}{t^{3/2}}\right), & t > 0, \end{cases} \tag{3.7}$$

and similarly

$$\begin{cases} \|P(t)\|_{L(BUC(\mathbb{R}^n), BUC^1(\mathbb{R}^n))} \le C\left(1 + \dfrac{1}{t^{1/2}}\right), & t > 0, \\[2ex] \|P(t)\|_{L(BUC(\mathbb{R}^n), BUC^3(\mathbb{R}^n))} \le C\left(1 + \dfrac{1}{t^{1/2}} + \dfrac{1}{t} + \dfrac{1}{t^{3/2}}\right), & t > 0. \end{cases} \tag{3.8}$$

Replacing $P(t)$ by $T(t) = P(t)e^{-t}$ (the semigroup generated by $A - I$) we get

$$\begin{cases} \|T(t)\|_{L(L^p, W^{1,p})} \le \dfrac{C}{t^{1/2}}, & t > 0, \\[2ex] \|T(t)\|_{L(L^p, W^{3,p})} \le \dfrac{C}{t^{3/2}}, & t > 0, \end{cases}$$

and

$$\begin{cases} \|T(t)\|_{L(BUC(\mathbb{R}^n), BUC^1(\mathbb{R}^n))} \le \dfrac{C}{t^{1/2}}, & t > 0, \\[2ex] \|T(t)\|_{L(BUC(\mathbb{R}^n), BUC^3(\mathbb{R}^n))} \le \dfrac{C}{t^{3/2}}, & t > 0. \end{cases}$$

Let $X = L^p(\mathbb{R}^n)$, $1 \le p < \infty$. Since $D(A) = D(A - I)$ then $D_A(\alpha/2, p) = D_{A-I}(\alpha/2, p)$. Using theorem 3.2.4, with $E = W^{1,p}(\mathbb{R}^n)$, $m = 1$, $\beta = 1/2$, we get

$$D_{A-I}(\alpha/2, p) = (X, D(A - I))_{\alpha/2,p} \subset (L^p(\mathbb{R}^n), W^{1,p}(\mathbb{R}^n))_{\alpha,p} = W^{\alpha,p}(\mathbb{R}^n).$$

Therefore,

$$W^{\alpha,p}(\mathbb{R}^n) \supset D_A(\alpha/2, p).$$

Moreover, since $D(A^2) = D((A - I)^2)$ then $D_A(\alpha/2 + 1, p) = D_{A-I}(\alpha/2 + 1, p)$. Using again theorem 3.2.4, with $E = W^{3,p}(\mathbb{R}^n)$, $m = 2$, $\beta = 3/4$, we get

$$D_{A-I}(\alpha/2 + 1, p) = (X, D((A - I)^2))_{(\alpha+2)/4,p}$$

$$\subset (L^p(\mathbb{R}^n), W^{3,p}(\mathbb{R}^n))_{(\alpha+2)/3,p} = W^{\alpha+2,p}(\mathbb{R}^n),$$

the last equality following from example 3.2.6. Therefore,

$$W^{\alpha+2,p}(\mathbb{R}^n) \supset D_A(\alpha/2 + 1, p),$$

and the first part of the statement is proved. The same procedure works in the case $X = BUC(\mathbb{R}^n)$, $X = C(\mathbb{R}^n)$, $X = L^\infty(\mathbb{R}^n)$. □

Remark 3.2.8 Note that the embeddings $\subset$ hold for every operator $A : D(A) \subset L^p(\mathbb{R}^n) \mapsto L^p(\mathbb{R}^n)$ such that $D(A) \supset W^{2,p}(\mathbb{R}^n)$ and $D(A^2) \supset W^{4,p}(\mathbb{R}^n)$ (respectively, $A : D(A) \subset BUC(\mathbb{R}^n) \mapsto BUC(\mathbb{R}^n)$ such that $D(A) \supset BUC^2(\mathbb{R}^n)$ and $D(A^2) \supset BUC^4(\mathbb{R}^n)$), whereas the embeddings $\supset$ hold for every operator $A : D(A) \subset L^p(\mathbb{R}^n) \mapsto L^p(\mathbb{R}^n)$ (respectively, $A : D(A) \subset BUC(\mathbb{R}^n) \mapsto BUC(\mathbb{R}^n)$) which generates a semigroup $P(t)$ satisfying estimates (3.7) (respectively, (3.8)). For $1 < p < \infty$ one could prove the statement also using the known characterizations $D(A) = W^{2,p}(\mathbb{R}^n)$, $D(A^2) = W^{4,p}(\mathbb{R}^n)$. However, such characterizations are not true for $p = 1$; similarly, it is not true that if $X = BUC(\mathbb{R}^n)$ then $D(A) = BUC^2(\mathbb{R}^n)$ and $D(A^2) = BUC^4(\mathbb{R}^n)$.

An important consequence of example 3.2.7 are the optimal regularity theorems for the Laplace equation in Hölder and in fractional Sobolev spaces.

Corollary 3.2.9 *(i) (Schauder Theorem) Let $u \in C_b^2(\mathbb{R}^n)$ be such that $\Delta u \in C_b^\alpha(\mathbb{R}^n)$ with $0 < \alpha < 1$. Then $u \in C_b^{\alpha+2}(\mathbb{R}^n)$, and*

$$\|u\|_{C_b^{\alpha+2}(\mathbb{R}^n)} \le C(\|u\|_\infty + \|\Delta u\|_{C_b^\alpha(\mathbb{R}^n)}).$$

(ii) Let $u \in W^{2,p}(\mathbb{R}^n)$ be such that $\Delta u \in W^{\alpha,p}(\mathbb{R}^n)$ with $0 < \alpha < 1$, $1 \le p < \infty$. Then $u \in W^{\alpha+2,p}(\mathbb{R}^n)$, and

$$\|u\|_{W^{\alpha+2,p}(\mathbb{R}^n)} \le C(\|u\|_{L^p} + \|\Delta u\|_{W^{\alpha,p}(\mathbb{R}^n)}).$$

Chapter 4

Powers of positive operators

The powers (with real or complex exponents) of positive operators are important tools in the study of partial differential equations. The theory of powers of operators is very close to interpolation theory, even if in general the domain of a power of a positive operator is not an interpolation space.

Through the whole chapter X is a complex Banach space.

4.1 Definitions and general properties

Definition 4.1.1 *A linear operator $A : D(A) \subset X \mapsto X$ is said to be a* positive operator *if the resolvent set of A contains $(-\infty, 0]$ and there is $M > 0$ such that*

$$\|R(\lambda, A)\|_{L(X)} \leq \frac{M}{1 + |\lambda|}, \quad \lambda \leq 0. \tag{4.1}$$

Note that A is a positive operator iff $(-\infty, 0)$ is a ray of minimal growth for the resolvent $R(\lambda, A)$ and $0 \in \rho(A)$. So, if A is a positive operator, then $-A$ satisfies (3.1) so that all the results of §3.1 are applicable.

Examples of unbounded positive operators are readily given: for instance, the realization of the first order derivative with Dirichlet boundary condition at $x = 0$ in $C([0, 1])$ or in $L^p(0, 1)$, $1 \leq p \leq \infty$ is positive. More generally, if A is the generator of a strongly continuous or analytic semigroup $T(t)$ such that $\|T(t)\| \leq Me^{-\omega t}$ for some $\omega > 0$, then $-A$ is a positive operator. This can be easily seen from the already mentioned resolvent formula

$$-R(-\lambda, -A) = R(\lambda, A) = \int_0^\infty e^{-\lambda t} T(t) dt, \quad \lambda > -\omega.$$

This section is devoted to the construction and to the main properties of the powers A^z, where z is an arbitrary complex number.

If $A : X \mapsto X$ is a bounded positive operator the powers A^z are readily defined by

$$A^z = \frac{1}{2\pi i} \int_\gamma \lambda^z R(\lambda, A) d\lambda,$$

where γ is any piecewise smooth curve surrounding $\sigma(A)$, avoiding $(-\infty, 0]$, with index 1 with respect to every element of $\sigma(A)$. Several properties of A^z follow easily from the definition: for instance, $z \mapsto A^z$ is holomorphic with values in $L(X)$; if $z = k \in \mathbb{Z}$ then

A^z defined above coincides with A^k; for each z, $w \in \mathbb{C}$ we have $A^z A^w = A^w A^z = A^{z+w}$; $(A^{-1})^z = A^{-z}$, etc.

In the case where A is unbounded the theory is much more complicated. To define A^z we shall use an elementary but important spectral property, stated in the next lemma.

Lemma 4.1.2 *Let A be a positive operator. Then the resolvent set of A contains the set*

$$\Lambda = \{\lambda \in \mathbb{C} : \text{ Re } \lambda \le 0, \ |\text{Im } \lambda| < (|\text{Re } \lambda| + 1)/M\} \cup \{\lambda \in \mathbb{C} : \ |\lambda| < 1/M\},$$

where M is the number in formula (4.1), and for every $\theta_0 \in (0, \arctan 1/M)$, $r_0 \in (0, 1/M)$ there is $M_0 > 0$ such that

$$\|R(\lambda, A)\| \le \frac{M_0}{1 + |\lambda|}$$

for all $\lambda \in \mathbb{C}$ with $|\lambda| \le r_0$, and for all $\lambda \in \mathbb{C}$ with $\text{Re } \lambda < 0$ *and* $|\text{Im } \lambda|/|\text{Re } \lambda| \le \tan \theta_0$.

Proof. It is sufficient to recall that for every $\lambda_0 \in \rho(A)$ the resolvent set $\rho(A)$ contains the open ball centered at λ_0 with radius $1/\|R(\lambda_0, A)\|$, and that for $|\lambda - \lambda_0| < 1/\|R(\lambda_0, A)\|$ it holds

$$R(\lambda, A) = \sum_{n=0}^{\infty} (-1)^n (\lambda - \lambda_0)^n R(\lambda_0, A)^{n+1}.$$

The union of the balls centered at $\lambda_0 \in (-\infty, 0]$ with radius $1/\|R(\lambda_0, A)\|$ contains the set Λ, and the estimate follows easily. □

For $\theta \in (\pi/2, \pi)$, $r > 0$, let $\gamma_{r,\theta}$ be the curve defined by $\gamma_{r,\theta} = -\gamma^{(1)}_{r,\theta} - \gamma^{(2)}_{r,\theta} + \gamma^{(3)}_{r,\theta}$, where $\gamma^{(1)}_{r,\theta}$, $\gamma^{(3)}_{r,\theta}$ are the half lines parametrized respectively by $z = \xi e^{i\theta}$, $z = \xi e^{-i\theta}$, $\xi \ge r$, and $\gamma^{(2)}_{r,\theta}$ is the arc of circle parametrized by $z = re^{i\eta}$, $-\theta \le \eta \le \theta$. See the figure.

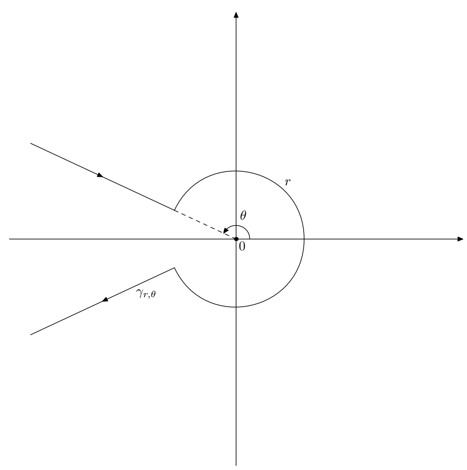

Fig. 1. The curve $\gamma_{r,\theta}$.

Now we are ready to define A^z for Re $z < 0$ through a Dunford integral.

Definition 4.1.3 *Fix any $r \in (0, 1/M)$, $\theta \in (\pi - \arctan 1/M, \pi)$. For* Re $\alpha < 0$ *set*

$$A^\alpha = \frac{1}{2\pi i} \int_{\gamma_{r,\theta}} \lambda^\alpha R(\lambda, A) d\lambda. \tag{4.2}$$

Since $\lambda \mapsto \lambda^\alpha R(\lambda, A)$ is holomorphic in $\Lambda \setminus (-\infty, 0]$ with values in $L(X)$, the integral is an element of $L(X)$ independent of r and θ. Writing down the integral we get

$$\begin{aligned} A^\alpha = & \frac{1}{2\pi i}\int_r^\infty \xi^\alpha(-e^{i\theta(\alpha+1)}R(\xi e^{i\theta}, A) + e^{-i\theta(\alpha+1)}R(\xi e^{-i\theta}, A))d\xi \\ & -\frac{r^{\alpha+1}}{2\pi}\int_{-\theta}^{\theta} e^{i\eta(\alpha+1)}R(re^{i\eta}, A)d\eta \end{aligned} \tag{4.3}$$

for every $r \in (0, 1/M)$, $\theta \in (\pi - \arctan 1/M, \pi)$.

Of course formula (4.3) may be reworked to get simpler expressions for A^α. For instance, if $-1 < \mathrm{Re}\,\alpha < 0$ we may let $r \to 0$, $\theta \to \pi$ to get

$$A^\alpha x = -\frac{\sin(\pi\alpha)}{\pi}\int_0^\infty \xi^\alpha(\xi I + A)^{-1}x\, d\xi. \tag{4.4}$$

Note that for every $a > 0$ and $\alpha \in (-1, 0)$ we have

$$a^\alpha = -\frac{\sin(\pi\alpha)}{\pi}\int_0^\infty \frac{\xi^\alpha}{\xi + a}\, d\xi, \tag{4.5}$$

which agrees with (4.4) of course, and will be used later.

From the definition it follows immediately that the function $z \mapsto A^z$ is holomorphic in the half plane $\mathrm{Re}\, z < 0$, with values in $L(X)$. Its behavior near the imaginary axis is not obvious, but it is of great importance in the developements of the theory and will be discussed in the next section.

Let us see some basic properties of the operators A^α.

Proposition 4.1.4 *The following statements hold true.*

(i) For $\alpha = -n$, $n \in \mathbb{N}$, the operator defined in (4.2) coincides with A^{-n}= n-th power of the inverse of A.

(ii) For Re $z < -k$, $k \in \mathbb{N}$, *the range of A^z is contained in the domain $D(A^k)$, and*

$$A^k A^z x = A^{k+z}x, \;\; x \in X.$$

(iii) For Re $z < 0$ *and $x \in D(A^k)$, $k \in \mathbb{N}$, $A^z x \in D(A^k)$, and*

$$A^z A^k x = A^k A^z x.$$

(iv) For Re z_1, Re $z_2 < 0$ *we have*

$$A^{z_1}A^{z_2} = A^{z_1+z_2}.$$

Proof. (i) Let $\alpha = -n$. It is easy to see that

$$\frac{1}{2\pi i}\int_{\gamma_{r,\theta}} \lambda^{-n}R(\lambda, A)d\lambda = \lim_{k\to\infty}\frac{1}{2\pi i}\int_{\gamma_k}\lambda^{-n}R(\lambda, A)d\lambda,$$

with γ_k as in figure.

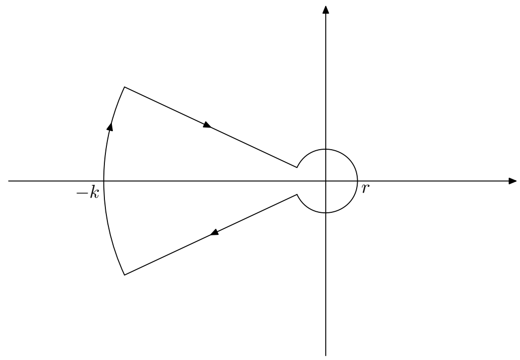

Fig. 2. The curve γ_k.

For every $k \in \mathbb{N}$ the function $\lambda \mapsto R(\lambda, A)$ is holomorphic in the bounded region surrounded by γ_k. For every $k \in \mathbb{N}$ we have

$$\frac{1}{2\pi i}\int_{\gamma_k} \lambda^{-n} R(\lambda, A) d\lambda = -\frac{1}{(n-1)!}\frac{d^{n-1}}{d\lambda^{n-1}} R(\lambda, A)_{|\lambda=0} = A^{-n},$$

and letting $k \to \infty$,

$$\frac{1}{2\pi i}\int_{\gamma_{r,\,\theta}} \lambda^{-n} R(\lambda, A) d\lambda = A^{-n}.$$

(ii) Let $k = 1$, Re $z < -1$. Then, since

$$\|\lambda^z A R(\lambda, A)\| = \|\lambda^z(\lambda R(\lambda, A) - I)\| \le (M_0 + 1)|\lambda|^{\mathrm{Re}\, z},$$

the integral defining A^z is in fact an element of $L(X, D(A))$, and

$$A\,\frac{1}{2\pi i}\int_{\gamma_{r,\theta}} \lambda^z R(\lambda, A) d\lambda = \frac{1}{2\pi i}\int_{\gamma_{r,\theta}} \lambda^{z+1} R(\lambda, A) d\lambda - \frac{1}{2\pi i}\int_{\gamma_{r,\theta}} \lambda^z d\lambda I.$$

But the last integral vanishes, so that $A \cdot A^z = A^{1+z}$, and the statement is proved for $k = 1$. The statement for any k follows arguing by recurrence.

Statement (iii) is obvious because A^k commutes with $R(\lambda, A)$ on $D(A^k)$, and this implies that A^k commutes with A^z on $D(A^k)$.

(iv) Let $\theta_1 < \theta_2 < \pi$, $1/M > r_1 > r_2 > 0$, so that γ_{r_1,θ_1} is on the right hand side of γ_{r_2,θ_2}. Then

$$A^{z_1}A^{z_2} = \frac{1}{(2\pi i)^2}\int_{\gamma_{r_1,\theta_1}} \lambda^{z_1} R(\lambda, A) d\lambda \int_{\gamma_{r_2,\theta_2}} w^{z_2} R(w, A) dw$$

$$= \frac{1}{(2\pi i)^2}\int_{\gamma_{r_1,\theta_1}\times\gamma_{r_2,\theta_2}} \lambda^{z_1} w^{z_2}\, \frac{R(\lambda, A) - R(w, A)}{w - \lambda} d\lambda\, dw$$

$$= \frac{1}{(2\pi i)^2}\int_{\gamma_{r_1,\theta_1}} \lambda^{z_1} R(\lambda, A) d\lambda \int_{\gamma_{r_2,\theta_2}} \frac{w^{z_2}}{w - \lambda}\, dw$$

$$-\frac{1}{(2\pi i)^2}\int_{\gamma_{r_2,\theta_2}} w^{z_2} R(w, A) dw \int_{\gamma_{r_1,\theta_1}} \frac{\lambda^{z_1}}{w - \lambda}\, d\lambda$$

$$= \frac{1}{2\pi i}\int_{\gamma_{r_1,\theta_1}} \lambda^{z_1+z_2} R(\lambda, A) d\lambda = A^{z_1+z_2}.$$

□

Statement (iv) of the proposition implies immediately that A^z is one to one. Indeed, if $A^z x = 0$ and $n \in \mathbb{N}$ is such that $-n < \text{Re } z$, then $A^{-n}x = A^{-n-z}A^z x = 0$, so that $x = 0$. Therefore it is possible to define A^α if Re $\alpha > 0$ as the inverse of $A^{-\alpha}$. But in this way the powers A^{it}, $t \in \mathbb{R}$, remain undefined. So we give a unified definition for Re $\alpha \geq 0$.

Definition 4.1.5 *Let* $0 \leq \text{Re } \alpha < n$, $n \in \mathbb{N}$. *We set*

$$D(A^\alpha) = \{x \in X : \ A^{\alpha-n}x \in D(A^n)\}, \ \ A^\alpha x = A^n A^{\alpha-n}x.$$

From proposition 4.1.4 it follows that the operator A^α is independent of n: indeed, if n, $m >$ Re α, then $A^{\alpha-m}x = A^{n-m}A^{\alpha-n}x$ both for $n < m$ (by proposition 4.1.4(iv)) and for $n > m$ (by proposition 4.1.4(ii), taking $z = \alpha - n$ and $k = n - m$), so that $A^{\alpha-m}x \in D(A^m)$ iff $A^{n-m}A^{\alpha-n}x \in D(A^m)$ i.e. $A^{\alpha-n}x \in D(A^n)$.

For $\alpha = 0$ we get immediately $A^0 = I$. Moreover for Re $\alpha > 0$ we get

$$D(A^\alpha) = A^{-\alpha}(X); \ \ A^\alpha = (A^{-\alpha})^{-1}.$$

Indeed, $A^{\alpha-n}x \in D(A^n)$ iff there is $y \in X$ with $A^{\alpha-n}x = A^{-n}y$. Such a y is obviously unique, and $A^\alpha x = y$ by definition. Moreover $A^{-n}A^{-\alpha}y = A^{-\alpha}A^{-n}y = A^{-\alpha}A^{\alpha-n}x = A^{-n}x$ so that $x = A^{-\alpha}y$ is in the range of $A^{-\alpha}$ and $A^\alpha = (A^{-\alpha})^{-1}$.

Since A^α has a bounded inverse, then it is a closed operator, so that $D(A^\alpha)$ is a Banach space endowed with the graph norm. Again, since A^α has a bounded inverse, its graph norm is equivalent to

$$x \mapsto \|A^\alpha x\|,$$

which is usually considered the canonical norm of $D(A^\alpha)$.

If Re $\alpha = 0$, $\alpha = it$ with $t \in \mathbb{R}$, A^{it} is the inverse of A^{-it} in the sense that for each $x \in D(A^{it})$, $A^{it}x \in D(A^{-it})$ and $A^{-it}A^{it}x = x$. Indeed, if $x \in D(A^{it})$ then $A^{it-1}x \in D(A)$, and $A^{it}x = A(A^{it-1}x)$ by definition. Therefore $A^{-1-it}A^{it}x = A^{-1-it}A \cdot A^{it-1}x = A \cdot A^{-1-it}A^{it-1}x = A \cdot A^{-2}x \in D(A)$, which implies that $A^{it}x \in D(A^{-it})$ and $A^{-it}A^{it}x = x$.

But in general the operators A^{it} are not bounded, see next example 4.2.1. However, they are closed operators, because A^{-1+it} is bounded and A is closed (see next exercise 6, §4.2.1). Therefore also $D(A^{it})$ is a Banach space under the graph norm.

From the definition it follows easily that for $0 \leq \text{Re } \alpha < n \in \mathbb{N}$, the domain $D(A^n)$ is continuously embedded in $D(A^\alpha)$: indeed for each $x \in D(A^n)$, $A^{\alpha-n}x \in D(A^n)$ by proposition 4.1.4(iii), and $A^\alpha x = A^n A^{\alpha-n}x = A^{\alpha-n}A^n x$ so that $\|A^\alpha x\| \leq \|A^{\alpha-n}\| \, \|A^n x\|$. This property is generalized in the next theorem.

Theorem 4.1.6 *Let* α, $\beta \in \mathbb{C}$ *be such that* Re $\beta <$ Re α. *Then* $D(A^\alpha) \subset D(A^\beta)$, *and for every* $x \in D(A^\alpha)$,

$$A^\beta x = A^{\beta-\alpha}A^\alpha x.$$

Moreover for each $x \in D(A^\alpha)$, $A^\beta x \in D(A^{\alpha-\beta})$ *and*

$$A^{\alpha-\beta}A^\beta x = A^\alpha x.$$

Conversely, if $x \in D(A^\beta)$ *and* $A^\beta x \in D(A^{\alpha-\beta})$, *then* $x \in D(A^\alpha)$ *and again* $A^{\alpha-\beta}A^\beta x = A^\alpha x$.

Proof. The embedding $D(A^\alpha) \subset D(A^\beta)$ is obvious if Re $\beta < 0$; it has to be proved for Re $\beta \geq 0$.

If $x \in D(A^\alpha)$, $A^{-n+\alpha}x \in D(A^n)$ for $n >$ Re α. Therefore $A^{-n+\beta}x = A^{\beta-\alpha}A^{-n+\alpha}x \in D(A^n)$, thanks to proposition 4.1.4(iii), so that $x \in D(A^\beta)$, and $A^\beta x = A^n A^{\beta-\alpha}A^{-n+\alpha}x = A^{\beta-\alpha}A^\alpha x$. Since $A^{\beta-\alpha}$ is a bounded operator, $\|A^\beta x\| \leq \|A^{\beta-\alpha}\|_{L(X)}\|A^\alpha x\|$, and $D(A^\alpha)$ is continuously embedded in $D(A^\beta)$.

Let again $x \in D(A^\alpha)$, and let $n > \max\{\mathrm{Re}\,\alpha, \mathrm{Re}\,(\alpha-\beta)\}$. Then

$$A^{-n+\alpha-\beta}A^\beta x = A^{-n+\alpha-\beta}A^{\beta-\alpha}A^\alpha x = A^{-n}A^\alpha x \in D(A^n),$$

so that $A^\beta x \in D(A^{\alpha-\beta})$ and $A^{\alpha-\beta}A^\beta x = A^\alpha x$.

Let now $x \in D(A^\beta)$ be such that $A^\beta x \in D(A^{\alpha-\beta})$, and fix $n > \max\{$ Re α, Re $\alpha - \beta\}$. Then

$$A^{\alpha-2n}x = A^{\alpha-n-\beta}A^{-n+\beta}x = A^{\alpha-n-\beta}A^{-n}A^\beta x = A^{-n}A^{\alpha-n-\beta}A^\beta x$$

is in $D(A^{2n})$, so that $x \in D(A^\alpha)$ and $A^\alpha x = A^{2n}A^{\alpha-2n}x = A^{\alpha-\beta}A^\beta x$. □

The condition Re $\beta <$ Re α is essential in the above theorem when Re $\alpha > 0$. In fact for every $\alpha > 0$, $t \in \mathbb{R}$ we have $D(A^\alpha) = D(A^{\alpha+it})$ if and only if A^{it} is bounded. See exercise 2, §4.2.1.

Now we give some representation formulas for $A^\alpha x$ when $x \in D(A^\alpha)$. We consider first the case where $0 <$ Re $\alpha < 1$. Taking $n = 1$ in the definition, we see that $x \in D(A^\alpha)$ if and only if $A^{\alpha-1}x \in D(A)$. Letting $r \to 0$ and $\theta \to \pi$ in the representation formula (4.3) for $A^{\alpha-1}x$ (i.e., using formula (4.4) with α replaced by $\alpha - 1$) we get

$$A^{\alpha-1}x = \frac{\sin(\pi\alpha)}{\pi}\int_0^\infty \xi^{\alpha-1}(\xi I + A)^{-1}x\,d\xi. \tag{4.6}$$

Therefore $x \in D(A^\alpha)$ if and only if the integral $\int_0^\infty \xi^{\alpha-1}(\xi I + A)^{-1}x\,d\xi$ is in the domain of A, and in this case

$$\begin{aligned} A^\alpha x &= \frac{\sin(\pi\alpha)}{\pi}\,A\int_0^\infty \xi^{\alpha-1}(\xi I + A)^{-1}x\,d\xi \\ &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\,A\int_0^\infty \xi^{\alpha-1}(\xi I + A)^{-1}x\,d\xi, \end{aligned} \tag{4.7}$$

which is the well-known Balakrishnan formula.

Another important representation formula holds for $-1 <$ Re $\alpha < 1$. The starting point is again formula (4.3) for $A^{\alpha-1}x$. We let $\theta \to \pi$ and then we integrate by parts in the integrals between r and ∞, getting

$$A^{\alpha-1}x = \frac{\sin(\pi\alpha)}{\pi\alpha}\int_r^\infty \xi^\alpha(\xi I + A)^{-2}x\,d\xi - r^\alpha\frac{\sin(\pi\alpha)}{\pi\alpha}(rI + A)^{-1}x$$

$$-\frac{r^\alpha}{2\pi}\int_{-\pi}^{\pi} e^{i\eta\alpha}(re^{i\eta}I + A)^{-1}x\,d\eta$$

(with $(\sin(\pi\alpha))/(\pi\alpha)$ replaced by 1 if $\alpha = 0$) and letting $r \to 0$ we get (both for Re $\alpha \in (0,1)$ and for Re $\alpha \in (-1,0]$)

$$A^{\alpha-1}x = \frac{1}{\Gamma(1-\alpha)\Gamma(1+\alpha)}\int_0^\infty \xi^\alpha(\xi I + A)^{-2}x\,d\xi. \tag{4.8}$$

Therefore $x \in D(A^\alpha)$ if and only if the integral $\int_0^\infty \xi^\alpha(\xi I + A)^{-2}x\, d\xi$ is in the domain of A, and in this case

$$A^\alpha x = \frac{1}{\Gamma(1+\alpha)\Gamma(1-\alpha)}\, A\int_0^\infty \xi^\alpha(\xi I + A)^{-2}x\, d\xi. \tag{4.9}$$

The most general formula of this type may be found as usual in the book of Triebel: for $n \in \mathbb{N}\cup\{0\}$, $m \in \mathbb{N}$, $-n < \text{Re }\alpha < m-n$ we have

$$A^\alpha x = \frac{\Gamma(m)}{\Gamma(\alpha+n)\Gamma(m-n-\alpha)}\, A^{m-n}\int_0^\infty t^{\alpha+n-1}(tI+A)^{-m}x\, dt$$

for every $x \in D(A^\alpha)$. See [36, §1.5.1].

We already know that the domain $D(A)$ is continuously embedded in $D(A^\alpha)$ for Re $\alpha \in [0,1)$. With the aid of the representation formulas (4.7) and (4.9) we are able to prove more precise embedding properties of $D(A^\alpha)$.

Proposition 4.1.7 *For $0 < \text{Re }\alpha < 1$, $D(A^\alpha) \in J_{\text{Re}\,\alpha}(X, D(A)) \cap K_{\text{Re}\,\alpha}(X, D(A))$, i.e.*

$$(X, D(A))_{\text{Re}\,\alpha,1} \subset D(A^\alpha) \subset (X, D(A))_{\text{Re}\,\alpha,\infty}.$$

Proof. The embedding $(X, D(A))_{\text{Re}\,\alpha,1} \subset D(A^\alpha)$ is easy, because for $\xi > 0$

$$\|A\xi^{\alpha-1}(\xi I + A)^{-1}x\| = \xi^{\text{Re}\,\alpha-1}\|A(\xi I + A)^{-1}x\|$$

and for every $x \in (X, D(A))_{\text{Re}\,\alpha,1}$ the function $\xi \mapsto \xi^{\text{Re}\,\alpha}\|A(A+\xi I)^{-1}x\|$ is in $L^1_*(0,\infty)$ thanks to proposition 3.1.1. Using the representation formula (4.6) for $A^{\alpha-1}x$, we get $A^{\alpha-1}x \in D(A)$, i.e. $x \in D(A^\alpha)$ and by (4.7)

$$\|A^\alpha x\| \le \frac{1}{|\Gamma(\alpha)\Gamma(1-\alpha)|}\int_0^\infty \xi^{\text{Re}\,\alpha}\|A(A+\xi I)^{-1}x\|\,\frac{d\xi}{\xi} \le C\|x\|_{(X,D(A))_{\text{Re}\,\alpha,1}}.$$

Let now $x \in D(A^\alpha)$. Then $x = A^{-\alpha}y$, with $y = A^\alpha x$, so that $x = A\cdot A^{-\alpha-1}y$. We use the representation formula (4.8) for $A^{-\alpha-1}y$, that gives

$$x = \frac{A}{\Gamma(1-\alpha)\Gamma(1+\alpha)}\int_0^\infty t^{-\alpha}(A+tI)^{-2}y\, dt.$$

On the other hand, by proposition 3.1.1 we have

$$\|x\|_{(X,D(A))_{\text{Re}\,\alpha,\infty}} \le C(\alpha)\sup_{\lambda>0}\|\lambda^{\text{Re}\,\alpha}A(A+\lambda I)^{-1}x\|,$$

so that

$$\|x\|_{(X,D(A))_{\text{Re}\,\alpha,\infty}} \le C(\alpha)\sup\nolimits_{\lambda>0}\left\|\frac{\lambda^{\text{Re}\,\alpha}A^2(A+\lambda I)^{-1}}{\Gamma(1-\alpha)\Gamma(1+\alpha)}\int_0^\infty t^{-\alpha}(A+tI)^{-2}y\, dt\right\|.$$

For every $\lambda > 0$ we have

$$\begin{aligned}
&\lambda^{\text{Re}\,\alpha}\left\|A^2(A+\lambda I)^{-1}\int_0^\infty t^{-\alpha}(A+tI)^{-2}y\, dt\right\| \\
&\le \lambda^{\text{Re}\,\alpha}\frac{M}{1+\lambda}\int_0^\lambda t^{-\text{Re}\,\alpha}(M+1)^2\|y\|\, dt \\
&+\lambda^{\text{Re}\,\alpha}(M+1)\int_\lambda^\infty t^{-\text{Re}\,\alpha}\frac{M(M+1)}{1+t}\,\|y\|dt \\
&\le C\|y\|
\end{aligned}$$

so that $x \in (X, D(A))_{\mathrm{Re}\,\alpha,\infty}$ and

$$\|x\|_{(X,D(A))_{\mathrm{Re}\,\alpha,\infty}} \le C'\|y\| = C'\|A^\alpha x\|,$$

which implies that $D(A^\alpha) \subset (X, D(A))_{\mathrm{Re}\,\alpha,\infty}$. □

Remark 4.1.8 Arguing similarly (using formula (4.9) instead of (4.7)) we see easily that for every $\varepsilon \in (0,1)$ and $t \in \mathbb{R}$, $(X, D(A))_{\varepsilon,1}$ is contained in $D(A^{it})$. Indeed the function $\xi \mapsto \|\xi^{it}A(\xi I+A)^{-2}x\| \le M(1+\xi)^{-1}\|A(\xi I+A)^{-1}x\|$ is in $L^1(0,\infty)$ for every x in $(X, D(A))_{\varepsilon,1}$, so that the integral $\int_0^\infty \xi^{it}(\xi I+A)^{-2}x\,d\xi$ belongs to the domain of A. Therefore, for every $\varepsilon \in (0,1)$ and $p \in [1,\infty]$, $t \in \mathbb{R}$, $(X, D(A))_{\varepsilon,p}$ is continuously embedded in $D(A^{it})$ (because it is continuously embedded in $(X, D(A))_{\varepsilon/2,1}$).

Remark 4.1.9 Let $0 < \alpha < 1$. It is possible to show that in its turn A^α is a positive operator, and that

$$R(\lambda, A^\alpha) = \frac{1}{2\pi i}\int_{\gamma_{r,\theta}} \frac{R(z,A)}{\lambda - z^\alpha}\,dz, \quad \lambda \le 0. \tag{4.10}$$

(see exercise 5, §4.2.1). Using the above formula for the resolvent, one shows that $-A^\alpha$ is a sectorial operator. This may be surprising, since $-A$ is not necessarily sectorial. This also may help in avoiding mistakes driven by "intuition". Consider for instance the case where $X = L^2(0,\pi)$ and A is the realization of $-d^2/dx^2$ with Dirichlet boundary condition, i.e. $A : D(A) = H^2(0,\pi) \cap H^1_0(0,\pi)$, $Au = -u''$. One could think that $A^{1/2}$ is a realization of $i\,d/dx$ with some boundary condition, but this cannot be true because such operators are not sectorial. See next example 4.3.10.

4.1.1 Powers of nonnegative operators

A part of the theory of powers of positive operators may be extended to nonnegative operators.

Definition 4.1.10 *A linear operator $A : D(A) \subset X \mapsto X$ is said to be nonnegative if the resolvent set of A contains $(-\infty, 0)$ and there is $M > 0$ such that*

$$\|(\lambda I + A)^{-1}\| \le \frac{M}{\lambda}, \quad \lambda > 0.$$

In other words, A is a nonnegative operator iff $(-\infty,0)$ is a ray of minimal growth for the resolvent of A.

An important example of nonnegative operator is the realization A of $-\Delta$ (the Laplace operator) in $L^p(\mathbb{R}^n)$, $1 \le p \le \infty$. But A is not positive because $0 \in \sigma(A)$. However if $p < \infty$ then A is one to one. See exercise 13, §4.2.1.

If $0 \in \sigma(A)$ but A is one to one, it is still possible to define A^z for $-1 < \mathrm{Re}\ z < 1$.

Let $-1 < \mathrm{Re}\ z < 1$, $z \ne 0$, and define an operator B_z on $D(A) \cap R(A)$ by

$$\begin{aligned} B_z x = \frac{\sin(\pi z)}{\pi}\bigg(&\frac{x}{z} - \frac{A^{-1}x}{1+z} + \\ &+ \int_0^1 \xi^{z+1}(\xi I + A)^{-1}A^{-1}x\,d\xi + \int_1^\infty \xi^{z-1}(\xi I+A)^{-1}Ax\,d\xi\bigg) \end{aligned} \tag{4.11}$$

for each $x \in D(A) \cap R(A)$ (note that in the case where $0 \in \rho(A)$, $B_z x$ coincides with $A^z x$ since formula (4.11) is obtained easily from (4.9)). Then one checks that $B_z : D(A) \cap R(A) \mapsto H$ is closable, and defines A^z as the closure of B_z.

Another way to define A^α for $0 < \alpha < 1$, even if A is not one to one, is the following: for $\lambda > 0$ one defines $(\lambda I + A^\alpha)^{-1}$ by

$$R_\lambda = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty \frac{\xi^\alpha}{\lambda^2 + 2\lambda\xi^\alpha \cos(\pi\alpha) + \xi^{2\alpha}} (\xi I + A)^{-1} d\xi, \tag{4.12}$$

then one checks that R_λ is invertible for every $\lambda > 0$ and $R_\lambda R_\mu = (R_\mu - R_\lambda)/(\lambda - \mu)$. Therefore there exists a unique closed operator B such that $R_\lambda = (\lambda I + B)^{-1}$ for $\lambda > 0$, and we set $A^\alpha = B$. (Note that in the case where $0 \in \rho(A)$ the above formula for $(\lambda I + A^\alpha)^{-1}$ is correct because it is obtained from (4.10) letting $r \to 0$ and $\theta \to \pi$).

From the representation formula (4.12) it follows that

$$\lim_{\varepsilon \to 0} (\lambda I + (\varepsilon I + A)^\alpha)^{-1} = (\lambda I + A^\alpha)^{-1}, \ \text{ in } L(H),$$

which will be used in the proof of next lemma. In its turn, lemma 4.1.11 will be used in the proof of theorem 4.3.4.

Lemma 4.1.11 *Let $A : D(A) \subset X \mapsto X$ be any nonnegative densely defined operator. Then $D(A + \varepsilon I)^\alpha = D(A^\alpha)$ for each $\alpha \in [0, 1]$, and there is C independent of ε such that*

$$\|(A + \varepsilon I)^\alpha x - A^\alpha x\| \le C\varepsilon^\alpha \|x\|, \ \ x \in D(A^\alpha).$$

Proof. For $0 < \eta < \varepsilon$, $A + \varepsilon I$ and $A + \eta I$ are positive operators. Using the Balakrishnan formula for $0 < \alpha < 1$, we get

$$\begin{aligned} &(A + \varepsilon I)^\alpha x - (A + \eta I)^\alpha x = \\ &= \frac{\sin(\pi\alpha)}{\pi} \left((\varepsilon - \eta) \int_\delta^\infty \xi^\alpha ((\xi + \varepsilon) I + A)^{-1} ((\xi + \eta) I + A)^{-1} x d\xi \right. \\ &\left. + \int_0^\delta \xi^{\alpha - 1} ((A + \varepsilon I)((\xi + \varepsilon) I + A)^{-1} - (A + \eta I)((\xi + \eta) I + A)^{-1}) x \, d\xi \right) \end{aligned}$$

for every $x \in D(A)$ and $\delta > 0$. Therefore,

$$\begin{aligned} &\|(A + \varepsilon I)^\alpha x - (A + \eta I)^\alpha x\| \\ &\le \frac{\sin(\pi\alpha)}{\pi} \left((\varepsilon - \eta) M^2 \int_\delta^\infty \xi^{\alpha - 2} d\xi + 2(1 + M) \int_0^\delta \xi^{\alpha - 1} d\xi \right) \\ &= \frac{\sin(\pi\alpha)}{\pi} \left(\frac{\varepsilon - \eta}{1 - \alpha} M^2 \delta^{\alpha - 1} + \frac{2(1 + M)}{\alpha} \delta^\alpha \right) \|x\| \end{aligned}$$

for every $x \in D(A)$ and $\delta > 0$. Taking $\delta = (\varepsilon - \eta)$ we get

$$\|(A + \varepsilon I)^\alpha x - (A + \eta I)^\alpha x\| \le C(\alpha)(\varepsilon - \eta)^\alpha \|x\|.$$

Therefore, for every $x \in D(A)$ the function $\varepsilon \mapsto (A+\varepsilon I)^\alpha x$ is uniformly continuous, so that there exists the limit $\lim_{\varepsilon\to 0}(A+\varepsilon I)^\alpha x = Bx$. Letting $\eta \to 0$ in the above estimate we find

$$\|(A+\varepsilon I)^\alpha x - Bx\| \le C(\alpha)\varepsilon^\alpha \|x\|. \tag{4.13}$$

for each $\varepsilon > 0$, for each $x \in D(A)$. But $D(A)$ is a core of $(A+\varepsilon I)^\alpha$, that is the closure of the restriction of $(A+\varepsilon I)^\alpha$ to $D(A)$ is $(A+\varepsilon I)^\alpha$ itself: indeed, for every $y \in D((A+\varepsilon I)^\alpha)$ the sequence $y_n = n(nI+A)^{-1}y$ is in $D(A)$, $y_n \to y$ and $(A+\varepsilon I)^\alpha y_n = n(nI+A)^{-1}(A+\varepsilon I)^\alpha y \to (A+\varepsilon I)^\alpha y$ as $n\to\infty$, because $D(A)$ is dense. This implies that B is closable, its closure $\overline{B}$ has domain $D((A+\varepsilon I)^\alpha)$, and inequality (4.13) holds also for $\overline{B}$. So, $(A+\varepsilon I)^\alpha x \to \overline{B}x$ uniformly for $x \in D(\overline{B})$, $\|x\| \le 1$, and this implies that $(-\infty, 0) \subset \rho(\overline{B})$, and $(\lambda I + (A+\varepsilon I)^\alpha)^{-1} \to (\lambda I + \overline{B})^{-1}$ as $\varepsilon \to 0$ in $L(X)$. Since $(\lambda I + (A+\varepsilon I)^\alpha)^{-1} \to (\lambda I + A^\alpha)^{-1}$, then $\overline{B} = A^\alpha$. □

4.2 Operators with bounded imaginary powers

Let again A be a positive operator in a complex Banach space X. We know that the operators A^z are bounded if Re $z < 0$ and unbounded in general if Re $z > 0$ (this is because $D(A^z) \subset (X, D(A))_{\mathrm{Re}\, z, \infty}$ by proposition 4.1.7). Moreover remark 4.1.8 tells us that for every $t \in \mathbb{R}$, the domain $D(A^{it})$ is not very far from the underlying space X because all the interpolation spaces $(X, D(A))_{\varepsilon,p}$ are continuously embedded in $D(A^{it})$. So, it is natural to ask whether $D(A^{it}) = X$ and A^{it} is a bounded operator also for $t \neq 0$. The general answer is "no", as the following example shows.

Example 4.2.1 *Let S be the shift operator, $S(\xi_1, \xi_2, \xi_3, \ldots) = (0, \xi_1, \xi_2, \ldots)$, in $X = c_0 =$ the space of all complex valued sequences ξ_n such that $\lim_{n\to\infty} \xi_n = 0$, endowed with the sup norm, and set $A = (I-S)^{-1}$. Then A is a positive operator, and for every $t \neq 0$, A^{it} is unbounded.*

Proof. It is easily seen that the domain of A is the subset of c_0 consisting of the sequences $\xi = \{\xi_n\}$ such that $\sum_{n=1}^\infty \xi_n = 0$ (which is dense in c_0), and

$$A(\xi_1, \xi_2, \xi_3, \ldots) = (\xi_1, \xi_1+\xi_2, \xi_1+\xi_2+\xi_3, \ldots),$$

An easy computation shows that for $\lambda > 0$

$$(\lambda I + A)^{-1} = \frac{I}{\lambda+1} - \frac{1}{(\lambda+1)^2}\sum_{k=1}^\infty \left(\frac{\lambda}{\lambda+1}\right)^{k-1} S^k,$$

so that

$$\|(\lambda I + A)^{-1}\| \le \frac{1}{\lambda+1}\left(1 + \frac{1}{\lambda+1}\sum_{k=1}^\infty \left(\frac{\lambda}{\lambda+1}\right)^{k-1}\right) \le \frac{2}{\lambda+1},$$

which implies that A is a positive operator. Replacing in (4.2), for Re $\alpha < 0$, and then also for Re $\alpha = 0$, we get

$$A^\alpha = I + \alpha S + \frac{\alpha(\alpha+1)}{2}S^2 + \frac{\alpha(\alpha+1)(\alpha+2)}{3!}S^3 + \ldots$$

So, for $\alpha = it$ the n-th component of $A^{it}\xi$ is

$$\xi_n + it\,\xi_{n-1} + \frac{it(it+1)}{2}\,\xi_{n-2} + \ldots + \frac{it(it+1)\cdot \ldots \cdot (it+n-2)}{(n-1)!}\,\xi_1.$$

Fix any $n \in \mathbb{N}$ and define a sequence $\xi \in D(A)$ as follows:

$$\xi_n = 0,\ \xi_{n-1} = \frac{1}{i},\ \xi_{n-2} = \frac{1}{i(it+1)},\ \xi_{n-3} = \frac{2}{i(it+1)(it+2)}, \ldots,$$

$$\xi_1 = \frac{(n-2)!}{i(it+1)\cdot \ldots \cdot (it+n-2)},$$

while for $k > n$ ξ_k is arbitrary, subject only to $|\xi_k| \leq 1$ and $\sum_{k=1}^{\infty} \xi_k = 0$. Then we get

$$\|A^{it}\xi\| \geq |(A^{it}\xi)_n| = |t|\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-2}\right)$$

while the norm of ξ is 1. Letting $n \to \infty$ we obtain $\sup_{\xi \in D(A),\, \|\xi\|=1} \|A^{it}\xi\| = +\infty$, so that A^{it} is unbounded. □

Let us discuss the behavior of A^z for Re $z < 0$, z close to the imaginary axis. From the representation formula (4.4) we get easily, using (4.5),

$$\|A^z\| \leq \frac{M}{\pi}\,|\sin(\pi z)| \int_0^{\infty} \frac{\xi^{\mathrm{Re}\,z}}{\xi + 1}\, d\xi \leq M \frac{|\sin(\pi z)|}{|\sin(\pi \mathrm{Re}\,z)|}, \quad \mathrm{Re}\,z \in (-1,0). \tag{4.14}$$

In particular, for real $z = -\alpha$, with $0 < \alpha < 1$ we get $\|A^{-\alpha}\| \leq M$, so that $\|A^{-\alpha}\|$ is bounded on the real interval $(-1,0)$, and hence it is bounded on any real interval $(-a,0)$, $a > 0$. But in general $\|A^z\|$ may be unbounded in other subsets Ω of the left halfplane such that $\overline{\Omega} \cap i\mathbb{R} \neq \emptyset$. However, if the operator A^{it} is bounded for $t \in I \subset \mathbb{R}$ and $\|A^{it}\| \leq C$ for every $t \in I$, then for $z = -\alpha + it$ we have

$$\|A^{-\alpha+it}\| \leq \|A^{-\alpha}\|\,\|A^{it}\| \leq MC, \quad 0 < \alpha < 1/2,\ t \in I.$$

A sort of converse of the above considerations is in the next lemma. It gives a simple (but hard to be checked) sufficient condition for A^{it} to be bounded.

Lemma 4.2.2 *Let A be a positive operator with dense domain $D(A)$. Assume that there are a set $\Omega \subset \{z \in \mathbb{C} : \mathrm{Re}\ z < 0\}$ and a constant $C > 0$ such that $\overline{\Omega} \cap i\mathbb{R} \neq \emptyset$ and $\|A^z\| \leq C$ for $z \in \Omega$. Then for every $t \in \mathbb{R}$ such that $it \in \overline{\Omega}$, A^{it} is a bounded operator and $\|A^{it}\| \leq C$.*

Proof. For every $x \in D(A)$ the function $z \mapsto A^z x$ is continuous for Re $z < 1$, so that $\lim_{z \to it,\, z \in \Omega} A^z x = A^{it}x$. Since $D(A)$ is dense and $\|A^z\| \leq C$ for $z \in \Omega$, it follows that for every $x \in X$ there exists the limit $\lim_{z \to it,\, z \in \Omega} A^z x$. Denoting such a limit by Tx, we get $\|Tx\| \leq C\|x\|$. Then A^{it} is a closed operator which coincides with the bounded operator T on a dense subset. This implies that $A^{it} = T$ so that A^{it} is bounded. □

The most popular examples of positive unbounded operators with bounded imaginary powers are m-accretive positive operators in Hilbert spaces, which will be discussed in the next section. Self-adjoint positive operators belong to this class. Another interesting example is the following.

Example 4.2.3 *Let $A : D(A) \mapsto X$ be any positive operator, and let $0 < \theta < 1$, $1 \leq p \leq \infty$. Then the parts of A in $(X, D(A))_{\theta,p}$ and in $(X, D(A))_\theta$ have bounded imaginary powers.*

Proof. It is not hard to check (see exercise 1, §4.2.1) that the part of A in any of the above spaces is still a positive operator.

First we consider the case $p = \infty$. Let $A_\theta : D_A(\theta+1, \infty) \mapsto D_A(\theta, \infty)$ be the part of A in $D_A(\theta, \infty)$. We already know (remark 4.1.8) that $D_A(\theta, \infty) = (X, D(A))_{\theta,\infty}$ is contained in $D(A^{it})$. Therefore for every $x \in D_A(\theta, \infty)$, $A^{it-1}x$ belongs to the domain $D(A)$. To obtain an estimate for $\|A^{it}x\| = \|A(A^{it-1}x)\|$ we use the representation formula (4.8) for $A^{it-1}x$,

$$\begin{aligned} A^{it-1}x &= \frac{1}{\Gamma(1-it)\Gamma(1+it)} \int_0^\infty \xi^{it}(A + \xi I)^{-2} x \, d\xi \\ &= \frac{\sin(\pi i t)}{\pi i t} \int_0^\infty \xi^{it}(A + \xi I)^{-2} x \, d\xi, \end{aligned}$$

which gives

$$\begin{aligned} \|A^{it}x\| &\leq \frac{e^{\pi t} - e^{-\pi t}}{2\pi t} \int_0^\infty \frac{M}{\xi^\theta(1+\xi)} \|\xi^\theta A(A + \xi I)^{-1} x\| \, d\xi \\ &\leq C(\theta) \frac{e^{\pi t} - e^{-\pi t}}{t} \|x\|_{(X,D(A))_{\theta,\infty}}. \end{aligned}$$

We prove now that in fact $A^{it-1}x$ belongs to $D_A(\theta+1, \infty)$. We use again the representation formula (4.8) for $A^{it-1}x$, which implies that for every $\lambda > 0$

$$\begin{aligned} &\|\lambda^\theta A(\lambda I + A)^{-1} A(A^{it-1}x)\| \\ &\leq \frac{e^{\pi t} - e^{-\pi t}}{2\pi t} \left\| \lambda^\theta (\lambda I + A)^{-1} \int_0^\lambda \xi^{it-\theta} A(A + \xi I)^{-1} \xi^\theta A(A + \xi I)^{-1} x \, d\xi \right\| \\ &+ \frac{e^{\pi t} - e^{-\pi t}}{2\pi t} \left\| \lambda^\theta A(\lambda I + A)^{-1} \int_\lambda^\infty \xi^{it-\theta} (A + \xi I)^{-1} \xi^\theta A(A + \xi I)^{-1} x \, d\xi \right\| \\ &\leq \frac{e^{\pi t} - e^{-\pi t}}{2\pi t} \left(\frac{M\lambda^\theta}{\lambda + 1} \frac{(M+1)\lambda^{1-\theta}}{1-\theta} + (M+1)\lambda^\theta \frac{M\lambda^{-\theta}}{\theta} \right) \|x\|_{(X,D(A))_{\theta,\infty}} \\ &\leq C' \frac{e^{\pi t} - e^{-\pi t}}{t} \|x\|_{(X,D(A))_{\theta,\infty}}. \end{aligned}$$

Therefore, $A^{it-1}x$ belongs to $D_A(\theta + 1, \infty)$, which is the domain of A_θ in $D_A(\theta, \infty)$. It follows that x is in the domain of A_θ^{it}, and

$$\|A_\theta^{it} x\|_{(X,D(A))_{\theta,\infty}} \leq C'' \frac{e^{\pi t} - e^{-\pi t}}{t} \|x\|_{(X,D(A))_{\theta,\infty}}.$$

The rest of the statement follows by interpolation: knowing that for $0 < \theta_1 < \theta_2 < 1$ the part of A^{it} in $(X, D(A))_{\theta_1,\infty}$ and in $(X, D(A))_{\theta_2,\infty}$ is a bounded operator, from theorem 1.1.6 it follows that for every $\theta \in (\theta_1, \theta_2)$ the part of A^{it} in $(X, D(A))_{\theta,p}$ and in $(X, D(A))_\theta$ is a bounded operator. □

Another important example follows from the so called “transference principle”. See Coifman–Weiss [16].

Theorem 4.2.4 *Let (Ω,μ) be a σ-finite measure space, and let $1<p<\infty$. If A is a positive operator in $L^p(\Omega,\mu)$ such that $\|(\lambda I+A)^{-1}\|\le 1/\lambda$ and $(\lambda I+A)^{-1}$ is positivity preserving for $\lambda>0$ (i.e. $f(x)\ge 0$ a.e. implies $((\lambda I+A)^{-1}f)(x)\ge 0$ a.e), then the operators A^{it} are bounded in $L^p(\Omega,\mu)$, and there is $C>0$ such that*

$$\|A^{it}\|\le C(1+t^2)e^{\pi|t|/2},\ \ t\in\mathbb{R}.$$

Let us come back to the general theory. Due to theorem 4.1.6, if the operators A^{it} are bounded for any t in a small neighborhood of 0, then they are bounded for every $t\in\mathbb{R}$. Moreover, if $\|A^{it}\|\le C$ for $-\delta\le t\le\delta$, then there exists C', $\gamma>0$ such that $\|A^{it}\|\le C'e^{\gamma|t|}$ for every $t\in\mathbb{R}$.

Lemma 4.2.5 *Let A be a positive operator such that $A^{it}\in L(X)$ for every $t\in\mathbb{R}$, and $t\mapsto\|A^{it}\|$ is locally bounded. Then for every $x\in\overline{D(A)}$ the function $z\mapsto A^zx$ is continuous in the closed halfplane* $\mathrm{Re}\,z\le 0$.

Proof. If $x\in D(A)$ then $z\mapsto A^zx$ is holomorphic for $\mathrm{Re}\,z<1$, so that it is obviously continuous for $\mathrm{Re}\,z\le 0$. We have already remarked that (4.14) implies $\|A^\alpha\|\le M$ for $-1/2<\alpha<0$, so that for $z=\alpha+i\beta$, $-1/2<\alpha<0$, we get

$$\|A^z\|\le M\|A^{i\beta}\|.$$

In particular, for every $t\in\mathbb{R}$ and $r>0$ small enough the norm $\|A^z-A^{it}\|$ is bounded in the half circle $\{z:\ |z-it|\le r,\ \mathrm{Re}\,z\le 0\}$, by a constant independent of z. It follows that for every $x\in\overline{D(A)}$, $\lim_{z\to it}A^zx=A^{it}x$. □

Note that if $x\notin\overline{D(A)}$ the function $z\mapsto A^zx$ cannot be continuous in the halfplane $\mathrm{Re}\,z\le 0$. Indeed by proposition 4.1.7, $A^zx\in(X,D(A))_{-\mathrm{Re}\,z,\infty}$ for $-1<\mathrm{Re}\,z<0$, and $(X,D(A))_{-\mathrm{Re}\,z,\infty}$ is contained in $\overline{D(A)}$.

The family of operators $\{A^{it}:\ t\in\mathbb{R}\}$ plays an important role also in the interpolation properties of the domains $D(A^z)$.

Theorem 4.2.6 *Let A be a positive operator with dense domain such that for every $t\in\mathbb{R}$ $A^{it}\in L(X)$, and there are C, $\gamma>0$ such that*

$$\|A^{it}\|\le Ce^{\gamma|t|},\ \ t\in\mathbb{R}.$$

Then for $0\le\mathrm{Re}\,\alpha<\mathrm{Re}\,\beta$

$$[D(A^\alpha),D(A^\beta)]_\theta=D(A^{(1-\theta)\alpha+\theta\beta}).$$

Proof. Thanks to theorem 4.1.6 we may assume that $\alpha=0$ without loss of generality. Moreover since A^{it} is bounded for every $t\in\mathbb{R}$, then $D(A^\beta)=D(A^{\mathrm{Re}\,\beta})$ for $\mathrm{Re}\,\beta>0$. See exercise 2, §4.2.1. So we may also assume that $\beta\in(0,\infty)$.

Let $x\in D(A^{\theta\beta})$, and set

$$f(z)=e^{(z-\theta)^2}A^{-(z-\theta)\beta}x,\ \ 0\le\mathrm{Re}\,z\le 1.$$

Let us prove that $f \in \mathcal{F}(X, D(A^\beta))$. f is obviously holomorphic in the strip Re $z \in (0,1)$ and continuous up to Re $z = 1$ with values in X. Since $D(A)$ is dense in X, f is also continuous up to Re $z = 0$ with values in X. Indeed, $A^{-(z-\theta)\beta}x = A^{-z\beta}A^{\theta\beta}x$, and we know from lemma 4.2.5 that $w \mapsto A^w y$ is continuous with values in X for Re $w \le 0$ for every $y \in \overline{D(A)} = X$. Similarly, $t \mapsto f(1+it)$ is continuous with values in $D(A^\beta)$. f is also bounded, since

$$\|A^{-(z-\theta)\beta}x\| = \|A^{-\beta \mathrm{Im}\, z}A^{-\beta \mathrm{Re}\, z}A^{\theta\beta}x\| \le \|A^{-\beta \mathrm{Re}\, z}\|\; Ce^{\gamma\beta|\mathrm{Im}\, z|}\|A^{\theta\beta}z\|$$

Therefore, $f \in \mathcal{F}(X, D(A^\beta))$. Since $f(\theta) = x$, then $x \in [X, D(A^\beta)]_\theta$ and

$$\begin{aligned}\|x\|_{[X,D(A^\beta)]_\theta} &\le \max\{\sup_{t\in\mathbb{R}} \|e^{-t^2+\theta^2}A^{-(it-\theta)\beta}x\|,\\ &\qquad \sup_{t\in\mathbb{R}} \|e^{-t^2+(1-\theta)^2}A^{-(1+it-\theta)\beta}x\|_{D(A^\beta)}\}\\ &\le C'\|A^{\theta\beta}x\|.\end{aligned}$$

It follows that $D(A^{\theta\beta})$ is continuously embedded in $[X, D(A^\beta)]_\theta$.

Conversely, let $x \in D(A^\beta)$, and let $f \in \mathcal{F}(X, D(A^\beta))$ be such that $f(\theta) = x$.

The function

$$F(z) = e^{(z-\theta)^2}A^{z\beta}f(z),$$

is continuous with values in X both for Re $z = 0$ and for Re $z = 1$, and we have

$$\sup_{t\in\mathbb{R}} \|F(it)\| \le \sup_{t\in\mathbb{R}} e^{-t^2+\theta^2}Ce^{\gamma\beta|t|}\sup_{t\in\mathbb{R}} \|f(it)\| \le C'\|f\|_{\mathcal{F}(X,D(A^\beta))}, \tag{4.15}$$

$$\sup_{t\in\mathbb{R}} \|F(1+it)\| \le \sup_{t\in\mathbb{R}} e^{-t^2+(1-\theta)^2}Ce^{\gamma\beta|t|}\sup_{t\in\mathbb{R}} \|A^\beta f(1+it)\| \le C'\|f\|_{\mathcal{F}(X,D(A^\beta))}, \tag{4.16}$$

so that F is bounded with values in X for Re $z = 0$ and for Re $z = 1$. If F would be holomorphic in the interior of S and continuous in S, we could apply the maximum principle to get "$\|A^\theta x\| \le C'\|f\|_{\mathcal{F}(X,D(A^\beta))}$". But in general F is not even defined in the interior of S, because f has values in X and not in the domain of some power of A. So we have to modify this procedure.

By remark 2.1.5,

$$\|x\|_{[X,D(A^\beta)]_\theta} = \inf\{\|f\|_{\mathcal{F}(X,D(A^\beta))} :\; f \in \mathcal{V}(X, D(A^\beta)),\; f(\theta) = x\}.$$

So, let $f \in \mathcal{V}(X, D(A^\beta))$ be such that $f(\theta) = x$. The function

$$F(z) = e^{(z-\theta)^2}A^{z\beta}f(z),\;\; 0 \le \mathrm{Re}\, z \le 1,$$

is now well defined and holomorphic for Re $z \in (0,1)$, continuous with values in X up to Re $z = 0$, Re $z = 1$, and bounded with values in X. By the maximum principle (see exercise 1, §2.1.1),

$$\begin{aligned}&\|A^{\theta\beta}x\| = \|f(\theta)\| \le \max\{\sup_{t\in\mathbb{R}} \|F(it)\|,\; \sup_{t\in\mathbb{R}} \|F(1+it)\|\}\\ &\le C'\|f\|_{\mathcal{F}(X,D(A^\beta))},\end{aligned}$$

where the last inequality follows from estimates (4.15) and (4.16).

Taking the infimum over all the $f \in \mathcal{V}(X, D(A^\beta))$ such that $f(\theta) = x$ we get

$$\|A^{\theta\beta}x\| \le C'\|x\|_{[X,D(A^\beta)]_\theta},\;\; x \in D(A^\beta).$$

Since $D(A^\beta)$ is dense in $[X, D(A^\beta)]_\theta$ it follows that $[X, D(A^\beta)]_\theta$ is continuously embedded in $D(A^{\theta\beta})$. □

4.2.1 Exercises

In exercises 1–9, A is a positive operator in general Banach space X.

1) Let Y be any interpolation space between X and $D(A)$. Prove that the part of A in Y is a positive operator.

2) Let Re $\alpha > 0$, $t \in \mathbb{R}$. Show that $D(A^\alpha) = D(A^{\alpha+it})$ with equivalence of the norms if and only if A^{it} is bounded.

3) Prove that (4.8) holds for $-1 < \text{Re}\,\alpha < 1$.

4) Show that for Re $\alpha > 1$, Re $\alpha \notin \mathbb{N}$, $D(A^\alpha)$ belongs to $J_{\{\text{Re}\,\alpha\}}(D(A^{[\text{Re}\,\alpha]}), D(A^{[\text{Re}\,\alpha]+1}))$ and to $K_{\{\text{Re}\,\alpha\}}(D(A^{[\text{Re}\,\alpha]}), D(A^{[\text{Re}\,\alpha]+1}))$, where $\{\text{Re}\,\alpha\}$ and $[\text{Re}\,\alpha]$ are the fractional part and the integral part of Re α, respectively.

5) Prove that for every $\alpha > 0$, A^α is a positive operator, and that

$$R(\lambda, A^\alpha) = \frac{1}{2\pi i}\int_{\gamma_{r,\theta}} \frac{R(z,A)}{\lambda - z^\alpha}\,dz, \quad \lambda \leq 0.$$

6) Prove that if $\Lambda : D(\Lambda) \subset X \mapsto X$ is a linear closed operator, and $B : X \mapsto X$ is a linear bounded operator, then $C : D(C) = \{x \in X : Bx \in D(\Lambda)\} \mapsto X$, $Cx = \Lambda Bx$, is a closed operator. (This is used to check that A^{it} is a closed operator for each $t \in \mathbb{R}$).

7) Prove that if $t,\ s \in \mathbb{R}$ and $x \in D(A^{is}) \cap D(A^{i(t+s)})$ then $A^{is}x \in D(A^{it})$ and $A^{it}A^{is}x = A^{i(t+s)}x$.

8) Improve the estimate of proposition 4.2.3 showing that for each $\beta < \arctan 1/M$ there is C such that

$$\|A^{it}\| \leq Ce^{(\pi-\beta)|t|}, \quad t \in \mathbb{R}.$$

Hint: instead of using formula (4.9), modify formula (4.3) for $A^{it-1}x$ letting only $r \to 0$ and leaving $\theta < \arctan 1/M$ fixed.

9) Prove that A^{-1} is a nonnegative operator, and that for $0 < \text{Re}\,\alpha < 1$

$$A^{-\alpha} = (A^{-1})^\alpha.$$

10) Let A be a densely defined nonnegative operator such that $R(A)$ is dense in X. Show that $R(A) \cap D(A)$ is dense in X, and that the operators $B_z : D(A) \cap R(A) \mapsto X$ defined in (4.11) are closable.

11) Let A be a nonnegative one to one operator, and set $A_\varepsilon = (\varepsilon I + A)(\varepsilon A + I)^{-1}$ for $\varepsilon > 0$. Show that (i) $\rho(A_\varepsilon) \supset (-\infty, 0]$, (ii) $(\lambda I + A_\varepsilon)^{-1} \to (\lambda I + A)^{-1}$ for $\lambda > 0$, $A_\varepsilon x \to x$ for $x \in D(A)$, $A_\varepsilon^{-1}x \to A^{-1}x$ for $x \in R(A)$ as $\varepsilon \to 0$, (iii) $A_\varepsilon^z x \to A^z x$ for $x \in D(A) \cap R(A)$ as $\varepsilon \to 0$.

12) Let A be a positive operator in a Hilbert space H. Show that for each $\alpha \in \mathbb{C}$, $(A^*)^\alpha = (A^{\overline{\alpha}})^*$, so that if α is real, then $(A^*)^\alpha = (A^\alpha)^*$.

13) Prove that the realization A of $-\Delta$ in $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, is a nonnegative operator. Prove that if $p < \infty$ A is one to one, and that if $p = \infty$ the kernel of A consists of the constant functions.

Hint: denoting by $T(t)$ the Gauss-Weierstrass semigroup, use $\|D_iT(t)\|_{L(L^p)} \leq C/\sqrt{t}$ to show that if $\Delta f = 0$ then $D_i f = D_iT(t)f$ vanishes for every $i = 1, \ldots, n$, so that f is constant.

4.2.2 The sum of two operators with bounded imaginary powers

We consider now two positive operators, $A : D(A) \subset X \mapsto X$, $B : D(B) \subset X \mapsto X$, having bounded imaginary powers and such that

$$\|A^{it}\| \le Me^{\gamma_A|t|}, \quad \|B^{it}\| \le Me^{\gamma_B|t|}, \quad t \in \mathbb{R}. \tag{4.17}$$

We also assume that A and B commute in the resolvent sense,

$$R(\lambda, A)R(\mu, B) = R(\mu, B)R(\lambda, A), \quad \lambda,\ \mu \le 0. \tag{4.18}$$

Our assumptions imply immediately (through formula (4.2)) that $A^zB^w = B^wA^z$ for Re w, Re $z < 0$ and also (less immediately but easily) for Re w, Re $z \le 0$.

We shall study the closability and the invertibility of the operator $A + B$, following the approach of Dore and Venni [18].

Proposition 4.2.7 *Let A, B be positive operators satisfying (4.17) and (4.18). Assume in addition that $\gamma_A+\gamma_B < \pi$ and that $D(A)$ or $D(B)$ is dense in X. Then $A+B : D(A)\cap D(B)$ is closable, and its closure $\overline{A+B}$ is invertible with inverse S given by*

$$S = \frac{1}{2i}\int_{a-i\infty}^{a+i\infty} \frac{A^{-z}B^{z-1}}{\sin(\pi z)}\, dz = \frac{1}{2i}\int_{a-i\infty}^{a+i\infty} \frac{B^{z-1}A^{-z}}{\sin(\pi z)}\, dz \tag{4.19}$$

with any $a \in (0, 1)$. Moreover, S is a left inverse of $A + B$.

Proof. Since $\gamma_A + \gamma_B < \pi$ the norm of the operator $A^{-z}B^{z-1}/\sin(\pi z)$ in the integral decays exponentially as $|\text{Im } z| \to \infty$. Therefore S is a bounded operator, and since $z \mapsto A^{-z}B^{z-1}/\sin(\pi z)$ is holomorphic in the strip Re $z \in (0, 1)$, S is independent of $a \in (0, 1)$.

The proof is in three steps: (i) we show that S is a left inverse of $A+B$; (ii) we show that if $D(A)$ (resp. $D(B)$) is dense, then for every $\varepsilon \in (0, 1)$ S maps $D(A^{1-\varepsilon})$ (resp. $D(B^{1-\varepsilon})$) to $D(A)\cap D(B)$ and $(A+B)Sx = x$ for each $x \in D(A^{1-\varepsilon})$ (resp. $x \in D(B^{1-\varepsilon})$); (iii) using steps (i) and (ii) we show in a standard way that $A + B$ is closable and $S = \overline{A + B}^{-1}$.

(i) For every $x \in D(A) \cap D(B)$ it holds

$$S(Ax + Bx) = \frac{1}{2i}\int_{a-i\infty}^{a+i\infty} \left(\frac{B^{z-1}A^{1-z}x}{\sin(\pi z)} + \frac{A^{-z}B^zx}{\sin(\pi z)}\right)dz$$

$$= -\frac{1}{2i}\int_{a-1-i\infty}^{a-1+i\infty} \frac{B^zA^{-z}x}{\sin(\pi z)}\, dz + \frac{1}{2i}\int_{a-i\infty}^{a+i\infty} \frac{A^{-z}B^zx}{\sin(\pi z)}\, dz.$$

The function $g : \Omega = \{z \in \mathbb{C} : -1 < \text{Re } z < 1\} \mapsto X$ defined by

$$g(z) = \begin{cases} B^zA^{-z}x, & -1 < \text{Re } z \le 0, \\ \\ A^{-z}B^zx, & 0 \le \text{Re } z < 1, \end{cases}$$

is holomorphic in the strips $-1 <$ Re $z < 0$, $0 <$ Re $z < 1$ and continuous up to the imaginary axis (see exercise 2, §4.2.3). Therefore it is holomorphic in the whole strip Ω. It follows that $z \mapsto g(z)/\sin(\pi z)$ is holomorphic in $\Omega \setminus \{0\}$. Moreover it has a simple pole at $z = 0$, with residue x/π, and it decays exponentially, uniformly for $-1 < \text{Re } z < 1$, as $|\text{ Im } z| \to \infty$. It follows that

$$S(A+B)x = \pi \text{Res}\left(\frac{g(z)}{\sin(\pi z)}, 0\right) = x.$$

(ii) Assume for instance that that $D(A)$ is dense. Fix $x \in D(A^{1-\varepsilon})$, with $0 < \varepsilon < 1$ and choose $a = \varepsilon$ in the definition of S. The function $z \mapsto A^{1-z}B^{z-1}x/\sin(\pi z)$ is well defined because B^{z-1} maps $D(A^{1-\varepsilon}) = D(A^{1-z})$ into itself, and it is integrable over $\varepsilon + i\mathbb{R}$, since

$$\|A^{1-z}B^{z-1}x\| = \|A^{\varepsilon-z}B^{z-1}A^{1-\varepsilon}x\| \le Ce^{(\gamma_A+\gamma_B)|\text{Im } z|}\|A^{1-\varepsilon}x\|.$$

Therefore, $Sx \in D(A)$ and

$$ASx = \frac{1}{2i}\int_{a-i\infty}^{a+i\infty} \frac{A^{1-z}B^{z-1}x}{\sin(\pi z)}\, dz = \frac{1}{2i}\int_{a-i\infty}^{a+i\infty} \frac{B^{z-1}A^{1-z}x}{\sin(\pi z)}\, dz.$$

To show that Sx belongs to $D(B)$, we remark that the function $z \mapsto B^{z-1}A^{-z}x/\sin(\pi z)$ is holomorphic for $\varepsilon - 1 < \text{Re } z < 1$, $z \neq 0$, continuous up to $\text{Re } z = \varepsilon - 1$ (because $D(A)$ is dense, see lemma 4.2.5), and it decays exponentially as $|\text{Im } z| \to \infty$. Therefore, we may shift the vertical line $\text{Re } z = a$ to $\text{Re } z = \varepsilon - 1$ in the definition of S, to get

$$\begin{aligned} Sx &= \frac{1}{2i}\int_{\varepsilon-1-i\infty}^{\varepsilon-1+i\infty} \frac{B^{z-1}A^{-z}x}{\sin(\pi z)}\, dz + \pi\text{Res}\left(\frac{B^{z-1}A^{-z}x}{\sin(\pi z)}, 0\right) \\ &= \frac{1}{2i}\int_{\varepsilon-1-i\infty}^{\varepsilon-1+i\infty} \frac{B^{z-1}A^{-z}x}{\sin(\pi z)}\, dz + B^{-1}x. \end{aligned}$$

As easily seen, the integral defines an element of $D(B)$. Therefore $Sx \in D(B)$ and

$$\begin{aligned} BSx &= \frac{1}{2i}\int_{\varepsilon-1-i\infty}^{\varepsilon-1+i\infty} \frac{B^{z}A^{-z}x}{\sin(\pi z)}\, dz + x \\ &= -\frac{1}{2i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \frac{B^{z-1}A^{1-z}x}{\sin(\pi z)}\, dz + x = -ASx + x. \end{aligned}$$

(iii) Let us show that $A+B$ is closable. Let $x_n \in D(A) \cap D(B)$ be such that $x_n \to 0$, $(A+B)x_n \to y$ as $n \to \infty$. Since S is a left inverse of $A+B$ then

$$0 = \lim_{n\to\infty} x_n = \lim_{n\to\infty} S(A+B)x_n = Sy.$$

Since $A^{-1}y \in D(A) \subset D(A^{1-\varepsilon})$ for each $\varepsilon > 0$, then by step (ii) $SA^{-1}y \in D(A) \cap D(B)$ and

$$A^{-1}y = (A+B)SA^{-1}y = (A+B)A^{-1}Sy = 0,$$

so that $y = 0$ and $A+B$ is closable.

As a last step we prove that $\overline{A+B}$ is invertible and its inverse is S. If $x \in D(\overline{A+B})$ there is a sequence $x_n \in D(A) \cap D(B)$ such that $x_n \to x$ and $(A+B)x_n \to \overline{A+B}\,x$ as $n \to \infty$. Then

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} S(A+B)x_n = S\overline{A+B}x,$$

which means that S is a left inverse of $\overline{A+B}$. Now for $x \in X$ let $x_n \in D(A)$ be such that $x_n \to x$ as $n \to \infty$. By step (ii) $Sx_n \in D(A) \cap D(B)$, $\lim_{n\to\infty} Sx_n = Sx$ and $\lim_{n\to\infty}(A+B)Sx_n = \lim_{n\to\infty} x_n = x$. This implies that $Sx \in D(\overline{A+B})$ and $\overline{A+B}\,Sx = x$, so that S is also a right inverse of $\overline{A+B}$. □

Under the assumptions of proposition 4.2.7 the operator $A+B$ is not closed in general. If we knew that $A+B$ is closed, it would follow that $A+B$ is invertible with bounded inverse. In the next proposition we use this fact to get information on the resolvent set of a positive operator with bounded imaginary powers.

Proposition 4.2.8 *Let A be a positive operator with bounded imaginary powers, satisfying*

$$\|A^{it}\| \leq Ce^{\gamma_A|t|}, \quad t \in \mathbb{R}.$$

If $0 \leq \gamma_A < \pi$, the resolvent set of A contains the sector $\{\lambda \in \mathbb{C} : |\arg\,(\lambda - \pi)| < \pi - \gamma_A\}$.

Proof. We apply proposition 4.2.7 to the operators A, $B = -\lambda I$ for every λ in the sector. If $-\lambda = \rho e^{i\theta}$ with $\rho > 0$, $\theta \in (-\pi, \pi)$, then $\|B^{it}\| = \|(-\lambda)^{it} I\| = e^{-\theta t}$, so that $\gamma_B = \theta = \arg\,(-\lambda)$. Since $D(B) = X$ and $A + B = A - \lambda I$ is closed, the statement follows from proposition 4.2.7. □

Under a further suitable assumption on the space X, the conditions of proposition 4.2.7 are sufficient for $A+B$ to be closed. Such assumption has several equivalent formulations.

The "geometric" formulation is the following: there exists a symmetric function $\zeta : X \times X \mapsto \mathbb{R}$ which is convex in each variable, such that $\zeta(0,0) > 0$, and $\zeta(x+y) \leq \|x+y\|$ whenever $\|x\| \leq 1 \leq \|y\|$. In this case the space X is said to be ζ-convex.

The "probabilistic" formulation is the following: there are $p \in (1,\infty)$, $C > 0$ such that for every probability space $(\Omega, \mathcal{F}, P)$ and for every martingale $\{u_k : \Omega \mapsto X\}$ with respect to any filtration $\{\mathcal{F}_k\}$, for each choice of $\varepsilon_k \in \{-1, 1\}$, and for each $n \in \mathbb{N}$ we have

$$\|\sum_{k=0}^{n} \varepsilon_k (u_k - u_{k-1})\|_{L^p(\Omega;X)} \leq C \|\sum_{k=0}^{n} (u_k - u_{k-1})\|_{L^p(\Omega;X)}$$

(where we set $u_{-1} = 0$). In this case X is said to be a UMD space, or to have the property of unconditionality of martingale differences.

The formulation which is useful here is the following: for some $p \in (1,\infty)$ and $\varepsilon > 0$, the *truncated Hilbert transform*

$$(H_\varepsilon f)(t) = \frac{1}{\pi} \int_{|y|\geq\varepsilon} \frac{f(t-y)}{y}\, dy, \quad t \in \mathbb{R}$$

is a bounded operator from $L^p(\mathbb{R}; X)$ to itself. Then it is possible to show that this is true for every $p \in (1,\infty)$ and $\varepsilon > 0$. Moreover for each $f \in L^p(\mathbb{R}; X)$ there exists the limit

$$\lim_{\varepsilon\to 0} H_\varepsilon f$$

in $L^p(\mathbb{R}; X)$ and a.e. pointwise. Such a limit is denoted by Hf and it is called the *Hilbert transform* of f.

Equivalence of the above properties is not trivial. Bourgain [6] showed that any UMD space has the Hilbert transform property. The converse was proved by Burkholder [11], who also proved in [10] that X is ζ-convex iff it is a UMD space.

Theorem 4.2.9 *Let X be a ζ-convex space, and let A, B be densely defined*[(1)] *positive operators in X satisfying the assumptions of proposition 4.2.7. Then $A+B : D(A)\cap D(B) \mapsto X$ is closed, and $0 \in \rho(A+B)$.*

Proof. After proposition 4.2.7, we have only to show that S maps X into $D(A)\cap D(B)$. This obviously implies that S is a right inverse of $A+B$, since by proposition 4.2.7 S is a right inverse of $\overline{A+B}$. Again by proposition 4.2.7, S is a left inverse of $A+B$. Therefore S is the inverse of $A+B$ and $0 \in \rho(A+B)$.

Let us show that S maps X into $D(B)$. For $0 < \varepsilon < 1/2$ we have

$$Sx = \frac{1}{2i}\int_{\Gamma_\varepsilon} \frac{A^{-z}B^{z-1}x}{\sin(\pi z)}\,dz$$

where Γ_ε is the curve $\{is : |s| \geq \varepsilon\} \cup \{z \in \mathbb{C} : |z| = \varepsilon\}$, Re $z \geq 0$, oriented with increasing imaginary part. So,

$$\begin{aligned} Sx &= \frac{1}{2}\int_{|s|\geq\varepsilon} \frac{A^{-is}B^{is-1}x}{\sin(\pi is)}\,ds + \frac{1}{2}\int_{-\pi/2}^{\pi/2} \frac{\varepsilon e^{i\theta}}{\sin(\pi\varepsilon e^{i\theta})}\,A^{-\varepsilon e^{i\theta}}B^{\varepsilon e^{i\theta}-1}x\,d\theta \\ &= I_{1,\varepsilon} + I_{2,\varepsilon}. \end{aligned}$$

Since $D(A)$ is dense, $I_{2,\varepsilon}$ goes to $B^{-1}x/2$ as $\varepsilon \to 0$. Moreover $I_{1,\varepsilon}$ is in $D(B)$ for every ε. We reach our goal if we show that $I_{1,\varepsilon}$ converges in $D(B)$ as $\varepsilon \to 0$, i.e. if $BI_{1,\varepsilon}$ converges in X as $\varepsilon \to 0$. Indeed, in that case we have

$$Sx = B^{-1}x/2 + \lim_{\varepsilon\to 0} I_{1,\varepsilon} \in D(B).$$

Let us split $BI_{1,\varepsilon}$ into the sum

$$BI_{1,\varepsilon} = \frac{1}{2}\int_{|s|\geq 1} \frac{A^{-is}B^{is}x}{\sin(\pi is)}\,ds + \frac{1}{2}\int_{\varepsilon\leq|s|\leq 1} \frac{A^{-is}B^{is}x}{\pi is}\,ds$$

$$+\frac{1}{2}\int_{\varepsilon\leq|s|\leq 1} A^{-is}B^{is}x\left(\frac{1}{\sin(\pi is)} - \frac{1}{\pi is}\right)ds.$$

The first term is independent of ε, the third one is easily seen to converge as $\varepsilon \to 0$ because $1/(\sin(\pi is)) - 1/(\pi is)$ is bounded. The second term is equal to $1/(2i)H_\varepsilon f(0)$ where

$$f(s) = \chi_{(-1,1)}(s)A^{is}B^{-is}x, \quad s \in \mathbb{R}.$$

Since $f \in L^p(\mathbb{R};X)$ for each $p > 1$ and X is ζ-convex, the truncated Hilbert transform of f converges in X for almost every $t \in \mathbb{R}$. Let us prove that 0 is one of such t's. Fix once and for all $t \in (0,1)$ such that $H_\varepsilon f(t)$ converges. Then

$$\pi(H_\varepsilon f)(0) = A^{-it}B^{it}\int_{\varepsilon\leq|s|\leq 1} \frac{A^{i(t-s)}B^{i(s-t)}x}{s}\,ds = A^{-it}B^{it}\cdot$$

$$\cdot\left(\int_{|s|\geq\varepsilon} \frac{f(t-s)}{s}\,ds + \int_{-1}^{t-1} \frac{A^{i(t-s)}B^{i(s-t)}x}{s}\,ds - \int_{1}^{t+1} \frac{A^{i(t-s)}B^{i(s-t)}x}{s}\,ds\right)$$

[1]Every ζ-convex space X is reflexive, and therefore by [27] all positive operators in X are densely defined.

converges as $\varepsilon \to 0$. This concludes the proof that $Sx \in D(B)$. In the same way one shows that $Sx \in D(A)$, and the statement follows. □

The main application – at least, from our point of view – of the theorem of Dore and Venni is to evolution equations in a ζ-convex space X,

$$\begin{cases} u(t) + Au(t) = f(t), & 0 < t < T, \\ u(0) = 0, \end{cases} \tag{4.20}$$

with $f \in L^p(0,T;X)$ for some $p \in (1,\infty)$, $T > 0$. Here we assume that $A : D(A) \subset X \mapsto X$ is a positive operator having bounded imaginary powers, and

$$\|A^{is}\| \le Ce^{\gamma_A|s|}, \quad s \in \mathbb{R}.$$

Then it is not hard to see that the operator $\mathcal{A}$ defined by

$$\mathcal{A} : L^p(0,T;D(A)) \mapsto \mathcal{X} = L^p(0,T;X), \quad (\mathcal{A}f)(t) = Af(t)$$

is positive and has bounded imaginary powers, given of course by

$$(\mathcal{A}^{is}f)(t) = A^{is}f(t), \quad s \in \mathbb{R}, \ f \in \mathcal{X},$$

and

$$\|\mathcal{A}^{is}\| \le Ce^{\gamma_A|s|}, \quad s \in \mathbb{R}.$$

It is also possible to show that the operator

$$\mathcal{B} : D(\mathcal{B}) = \{f \in W^{1,p}(0,T;X) : \ f(0) = 0\} \mapsto \mathcal{X}, \quad (\mathcal{B}f)(t) = f(t),$$

is positive and has bounded imaginary powers, satisfying the estimate

$$\|\mathcal{B}^{is}\| \le C(1 + |s|^2)e^{\pi|s|/2}, \quad s \in \mathbb{R}.$$

See [18]. Therefore, if $\gamma_A < \pi/2$ it is possible to apply theorem 4.2.9 to equation (4.20), seen as the equation in $\mathcal{X}$

$$\mathcal{A}u + \mathcal{B}u = f,$$

getting that for every $f \in L^p(0,T;X)$ problem (4.20) has a unique solution $u \in W^{1,p}(0,T;X) \cap L^p(0,T;D(A))$, which depends continuously on f.

For instance, if A is the realization of $-\Delta$ in $L^p(\Omega)$, $1 < p < \infty$, with Dirichlet boundary condition:

$$D(A) = W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega), \quad Au = -\Delta u,$$

Ω being an open bounded set in $\mathbb{R}^n$ with regular boundary, then A is a positive operator with bounded imaginary powers, and $\gamma_A, \pi/2$ due to [34]. We get that for each $f \in L^p((0,T) \times \Omega)$ the problem

$$\begin{cases} u_t(t,x) = \Delta u(t,x) + f(t,x), & 0 < t < T, \ x \in \Omega, \\ u(t,x) = 0, & 0 < t < T, \ x \in \partial\Omega, \\ u(0,x) = 0, & x \in \Omega, \end{cases}$$

has a unique solution $u \in W^{1,2}_p((0,T) \times \Omega)$, i.e. u, u_t, $D_{ij}u \in L^p((0,T) \times \Omega)$.

4.2.3 Exercises

1) Let A, B be two positive operators with bounded imaginary powers, satisfying (4.18). Show that $A^z B^w = B^w A^z$ for Re z, Re $w \leq 0$. Show that for Re $z \leq 0$, A^z maps $D(B)$ into itself, and $A^z Bx = BA^z x$ for each $x \in D(B)$. Show that for Re $z \leq 0$, A^z maps $\overline{D(B)}$ into itself.

2) Let A, B be two positive operators with bounded imaginary powers, satisfying (4.17) and (4.18). Show that for every $x \in D(A) \cap D(B)$ the functions $z \mapsto B^z A^{-z} x$, $-1 <$ Re $z \leq 0$, and $z \mapsto A^{-z} B^z x$, $0 \leq$ Re $z < 1$, are continuous (this is used in the proof of proposition 4.2.7).

4.3 M-accretive operators in Hilbert spaces

Throughout this section H is a complex Hilbert space, and $A : D(A) \subset H \mapsto H$ is a linear operator satisfying

$$\overline{D(A)} = H, \ \ \rho(A) \supset (-\infty, 0), \ \ \|(\lambda I + A)^{-1}\| \leq 1/\lambda, \ \ \lambda > 0. \tag{4.21}$$

Therefore, A is a nonnegative operator. Moreover it satisfies the resolvent estimate with constant $M = 1$, and this is not a mere notational simplification but it is a crucial assumption.

Due to the Hille-Yosida theorem, assumption (4.21) is equivalent to the hypothesis that $-A$ be the infinitesimal generator of a contraction semigroup e^{-tA}. Therefore, for each $x \in D(A)$,

$$\text{Re}\,\langle -Ax, x\rangle = \lim_{t\to 0} \text{Re}\,\langle \frac{e^{-tA}x - x}{t}, x\rangle \leq 0.$$

Any operator $B : D(B) \subset H \mapsto H$ satisfing

$$\text{Re}\,\langle Bx, x\rangle \geq 0, \ \ x \in D(B)$$

is called *accretive*. Therefore, A is accretive.

It is possible to show that A is *m-accretive* (maximal accretive), in the sense that it has no proper accretive extension, and conversely, any closed m-accretive operator satisfies (4.21). So, operators satisfying (4.21) are often referred in the literature as m-accretive operators.

It is easy to see that A satisfies (4.21) iff A^* does. Moreover, if A satisfies (4.21) and it is one to one, then the range of A is dense in H.

We shall follow the approach of Kato [25, 26] to study the imaginary powers of A and the relationship between the domains of A^α and $(A^*)^\alpha$.

First we consider m-accretive bounded operators satisfying in addition

$$\text{Re}\,\langle Ax, x\rangle \geq \delta\|x\|^2, \ \ x \in H. \tag{4.22}$$

for some $\delta > 0$. The case of unbounded operators will be reduced to this one, through the use of the Yosida approximations $nA(nI + A)^{-1}$.

Assumption (4.22) implies that the resolvent set of A contains $(-\infty, \delta)$, and $\|(\lambda I + A)^{-1}\| \leq (\lambda + \delta)^{-1}$ for every $\lambda \geq 0$. Therefore A is a positive bounded operator, so that for each $z \in \mathbb{C}$ the complex powers A^z are defined by

$$A^z = \frac{1}{2\pi i}\int_\gamma \lambda^z R(\lambda, A)d\lambda, \tag{4.23}$$

where γ is any regular curve surrounding $\sigma(A)$ with index 1 with respect to every point of $\sigma(A)$, and avoiding $(-\infty, 0]$. Moreover we have

$$(A^\alpha)^* = (A^*)^{\overline{\alpha}}, \ \ \alpha \in \mathbb{C}.$$

See exercise 12, §4.2.1.

We will need the following lemma.

Lemma 4.3.1 *Let A be a bounded m-accretive operator satisfying (4.22). Then for real $\beta \in [-1, 1]$, A^β satisfies*

$$\text{Re}\ \langle A^\beta x, x\rangle \geq \delta^\beta \|x\|^2, \ \ 0 \leq \beta \leq 1, \ x \in H, \tag{4.24}$$

$$\text{Re}\ \langle A^\beta x, x\rangle \geq (\delta\|A\|^{-2})^\beta \|x\|^2, \ \ -1 \leq \beta \leq 0, \ x \in H. \tag{4.25}$$

Proof. For $0 < \beta < 1$ we use the Balakrishnan formula (4.7), which implies

$$\langle A^\beta x, x\rangle = \frac{\sin(\pi\beta)}{\pi} \int_0^\infty \xi^{\beta-1} \langle A(\xi I + A)^{-1} x, x\rangle d\xi.$$

Recalling that

$$\text{Re}\langle A(\xi I + A)^{-1} x, x\rangle = \text{Re}\langle (I - \xi(\xi I + A)^{-1})x, x\rangle$$

$$\geq \|x\|^2 - \xi|\langle(\xi I + A)^{-1}x, x\rangle| \geq \|x\|^2 - \frac{\xi}{\xi + \delta}\|x\|^2$$

$$= \frac{\delta}{\xi + \delta}\|x\|^2,$$

we obtain, through (4.5),

$$\langle A^\beta x, x\rangle \geq \frac{\delta \sin(\pi\beta)}{\pi} \int_0^\infty \frac{\xi^{\beta-1}}{\xi + \delta} \|x\|^2 d\xi = \delta^\beta \|x\|^2,$$

i.e. (4.24) holds. Moreover,

$$\text{Re}\langle A^{-1}x, x\rangle = \text{Re}\langle A^{-1}x, AA^{-1}x\rangle \geq \delta\|A^{-1}x\|^2 \geq \delta\|A\|^{-2}\|x\|^2,$$

so that (4.25) holds for $\beta = -1$. But using again the representation formula (4.23) we see easily that

$$A^\beta = (A^{-1})^{-\beta}, \ \ \beta \in \mathbb{C},$$

so that (4.25) holds for every $\beta \in [-1, 0]$. □

Theorem 4.3.2 *Let $A : H \mapsto H$ be a bounded operator. Assume that there exists $\delta > 0$ such that (4.22) holds. Then for each $\alpha \in [0, 1/2)$ and for each $x \in H$*

$$\|(A^*)^\alpha x\| \leq c_\alpha \|A^\alpha x\|, \ \ \|A^\alpha x\| \leq c_\alpha \|(A^*)^\alpha x\|,$$

with $c_\alpha = \tan \pi(1 + 2\alpha)/4$. Moreover,

$$\|A^{it}\| \leq e^{\pi|t|/2}, \ \ t \in \mathbb{R}.$$

Proof. Let us introduce the "real part" and the "imaginary part" of A^α defined by

$$H_\alpha = \frac{A^\alpha + (A^*)^\alpha}{2}, \;\; K_\alpha = \frac{A^\alpha - (A^*)^\alpha}{2i},$$

i.e.

$$A^\alpha = H_\alpha + iK_\alpha, \;\; (A^*)^\alpha = H_\alpha - iK_\alpha.$$

Then for every $x \in H$

$$\|H_\alpha x\|^2 - \|K_\alpha x\|^2 = \text{Re}\,\langle A^\alpha x, (A^*)^\alpha x\rangle = \text{Re}\,\langle A^{\alpha+\overline{\alpha}} x, x\rangle.$$

If $-1/2 \leq \text{Re}\;\alpha \leq 1/2$, then $A^{\alpha+\overline{\alpha}}$ still satisfies (4.22) with δ replaced by the constant $\delta_\alpha = \min\{\delta^\alpha, (\delta\|A\|^{-2})^\alpha\}$ thanks to lemma 4.3.1. Therefore,

$$\|H_\alpha x\|^2 \geq \|K_\alpha x\|^2 + \delta_\alpha \|x\|^2, \;\; x \in H. \tag{4.26}$$

This implies that H_α is one to one. Since $H_\alpha^* = (A^{\overline{\alpha}} + (A^*)^{\overline{\alpha}})/2$, and A^* satisfies the assumptions of the theorem, also H_α^* is one to one. Therefore H_α is invertible, and since $\|K_\alpha y\| \leq \|H_\alpha y\|$ for every y because of (4.26), then $\|K_\alpha H_\alpha^{-1} x\| \leq \|H_\alpha H_\alpha^{-1} x\| = \|x\|$ for each x, so that

$$\|K_\alpha H_\alpha^{-1}\| \leq 1, \;\; -1/2 \leq \text{Re}\;\alpha \leq 1/2.$$

Now we improve this estimate applying the maximum principle to the function

$$\alpha \mapsto \Phi(\alpha) = \frac{K_\alpha H_\alpha^{-1}}{\tan(\pi\alpha/2)}, \;\; -1/2 \leq \text{Re}\;\alpha \leq 1/2.$$

Such a function is holomorphic in the whole strip (even at $\alpha = 0$, because $K_0 = 0$) and bounded with values in $L(H)$; moreover $|\tan(\pi\alpha/2)| = 1$ if $\text{Re}\;\alpha = \pm 1/2$ so that $\|\Phi(\alpha)\| \leq 1$ on the boundary of the strip. Therefore $\|\Phi(\alpha)\| \leq 1$ for each α in the strip, i.e.

$$\|K_\alpha H_\alpha^{-1}\| \leq \tan(\pi\alpha/2), \;\; -1/2 \leq \text{Re}\;\alpha \leq 1/2. \tag{4.27}$$

This is an important improvement of $\|K_\alpha H_\alpha^{-1}\| \leq 1$, because $|\tan(\pi\alpha/2)| < 1$ for $|\text{Re}\;\alpha| < 1/2$, so that $I \pm iK_\alpha H_\alpha^{-1}$ is invertible with bounded inverse.

Since $A^\alpha = H_\alpha + iK_\alpha$, $(A^*)^\alpha = H_\alpha - iK_\alpha$, we get for $|\text{Re}\;\alpha| < 1/2$

$$\|(A^*)^\alpha A^{-\alpha}\| = \|(I - iK_\alpha H_\alpha^{-1})(I + iK_\alpha H_\alpha^{-1})^{-1}\| \leq \frac{1 + |\tan(\pi\alpha/2)|}{1 - |\tan(\pi\alpha/2)|}. \tag{4.28}$$

Therefore for real $\alpha \in [0, 1/2)$

$$\|(A^*)^\alpha x\| \leq \frac{1 + \tan(\pi\alpha/2)}{1 - \tan(\pi\alpha/2)}\|A^\alpha x\| = \tan\frac{\pi(1+2\alpha)}{4}\|A^\alpha x\|, \;\; x \in H,$$

and we get a similar inequality exchanging A with A^*. So, the first statement is proved.

Moreover, taking $\alpha = -it$ with real t, in (4.28) we get

$$\|A^{-it}x\|^2 = \langle (A^*)^{it} A^{-it} x, x\rangle \leq \frac{1 + |\tan(\pi it/2)|}{1 - |\tan(\pi it/2)|}\|x\|^2 = e^{\pi|t|}\|x\|^2,$$

and the last statement follows. □

Theorem 4.3.2 is the starting point to show several properties of the powers of general m-accretive operators. The first property concerns the equivalence of the domains $D(A^\alpha)$, $D((A^*)^\alpha)$ for $0 \leq \alpha < 1/2$. But we need a preliminary lemma.

Lemma 4.3.3 *Let $A : D(A) \subset H \mapsto H$ satisfy (4.21). For every $n \in \mathbb{N}$ set*

$$J_n = \left(I + \frac{A}{n}\right)^{-1} = n(nI + A)^{-1}.$$

Then J_n is a bounded nonnegative operator, and for every $\alpha \in [0, 1]$

$$\|J_n^\alpha\| \le 1,$$

$$\lim_{n\to\infty} J_n^\alpha x = x, \quad x \in H.$$

Proof. As easily seen, for every $\lambda > 0$ $\lambda I + J_n$ is invertible with (bounded) inverse

$$(\lambda I + J_n)^{-1} = (nI + A)(n(\lambda + 1) + \lambda A)^{-1}.$$

For every $\varepsilon > 0$ the operator $I + \varepsilon A$ is positive. The representation formula (4.6) gives

$$(I + A/n)^{-\alpha} = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty \xi^{-\alpha}(\xi I + I + A/n)^{-1} d\xi,$$

where $\|(\xi I + I + A/n)^{-1}\| \le 1/(\xi + 1)$, so that due to (4.5)

$$\|(I + \varepsilon A)^{-\alpha}\| \le \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty \frac{1}{\xi^{-\alpha}(\xi + 1)} d\xi = 1.$$

It follows (see exercise 9, §4.2.1)

$$\|(I + A/n)^{-\alpha}\| = \|((I + A/n)^{-1})^{-\alpha}\| = \|J_n^\alpha\| \le 1,$$

and by the dominated convergence theorem

$$\lim_{n\to\infty} J_n^\alpha x = \lim_{n\to\infty} (I + A/n)^{-\alpha} x = \lim_{n\to\infty} \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty \frac{\xi^{-\alpha}}{\xi + 1} x \, d\xi = x, \quad x \in H.$$

□

Theorem 4.3.4 *Let $A : D(A) \subset H \mapsto H$ be any m-accretive operator. Then for every $\alpha \in [0, 1/2)$, $D(A^\alpha) = D((A^*)^\alpha)$ and for each x in the common domain*

$$\|(A^*)^\alpha x\| \le \tan \frac{\pi(1 + 2\alpha)}{4} \|A^\alpha x\|, \quad \|A^\alpha x\| \le \tan \frac{\pi(1 + 2\alpha)}{4} \|(A^*)^\alpha x\|.$$

Proof. We know already (theorem 4.3.2) that the statement is true if A is bounded and satisfies (4.22) for some $\delta > 0$. As a second step, we prove that the statement is true if A satisfies (4.21) and $0 \in \rho(A)$. Then it will follow easily that the statement is true in the general case.

Let $0 \in \rho(A)$. Let us consider the Yosida approximations,

$$A_n = AJ_n = nA(nI + A)^{-1}, \quad n \in \mathbb{N}.$$

It is not hard to see that the operators A_n are bounded (with $\|A_n\| \le n$), m-accretive (with $(\xi I + A_n)^{-1} = n^2/(n + \xi)^2 (n\xi/(n + \xi) I + A)^{-1} + I/(n + \xi))$, and satisfy (4.22) since

$$\langle A_n x, x\rangle = \langle AJ_n x, \frac{1}{n}(nI + A)J_n x\rangle = \langle AJ_n x, J_n x\rangle + \frac{1}{n}\|A_n x\|^2$$

and A_n is invertible, with $A_n^{-1} = A^{-1} + I/n$. Therefore by theorem 4.3.2 we have

$$\begin{cases} \|(A_n^*)^\alpha\| \le \tan\dfrac{\pi(1+2\alpha)}{4}\|A_n^\alpha\|, \\[2ex] \|A_n^\alpha\| \le \tan\dfrac{\pi(1+2\alpha)}{4}\|(A_n^*)^\alpha\|, \end{cases} \tag{4.29}$$

for every $n \in \mathbb{N}$.

Note that since $I + A/n$ is a positive operator, then $(I + A/n)^{-\alpha}$ is well defined for every $\alpha \ge 0$, and by exercise 9, §4.2.1, $(I + A/n)^{-\alpha} = J_n^\alpha$ for $0 \le \alpha \le 1$. Therefore,

$$J_n^\alpha = (A^{-1}A_n)^\alpha = A_n^\alpha A^{-\alpha} = A^{-\alpha}A_n^\alpha,$$

as it is easy to check. Therefore for every $x \in D(A^\alpha)$, $J_n^\alpha x \in D(A^\alpha)$, and we have

$$A_n^\alpha x = A^\alpha J_n^\alpha x = J_n^\alpha A^\alpha x, \;\; 0 \le \alpha \le 1.$$

Now we use lemma 4.3.3, which states that $\|J_n^\alpha\| \le 1$, and $\lim_{n\to\infty} J_n^\alpha x = x$ for each x. We get

$$(i)\; \|A_n^\alpha x\| \le \|A^\alpha x\|, \;\; (ii)\; \lim_{n\to\infty} A_n^\alpha x = A^\alpha x, \;\; x \in D(A^\alpha). \tag{4.30}$$

Using (4.29) for $0 \le \alpha < 1/2$ and then (4.30)(i) we get

$$\|(A_n^*)^\alpha x\| \le \tan\frac{\pi(1+2\alpha)}{4}\|A_n^\alpha x\| \le \tan\frac{\pi(1+2\alpha)}{4}\|A^\alpha x\|, \tag{4.31}$$

so that the sequence $(A_n^*)^\alpha x$ is bounded for each $x \in D(A^\alpha)$. Moreover, for every $y \in D(A^\alpha)$ we have, due to (4.30)(ii),

$$\langle (A_n^*)^\alpha x, y\rangle = \langle x, A_n^\alpha y\rangle \to \langle x, A^\alpha y\rangle, \;\; n \to \infty.$$

Since $D(A^\alpha)$ is dense in H (because $D(A)$ is dense and $D(A^\alpha) \supset D(A)$), and $\|(A_n^*)^\alpha x\|$ is bounded thanks to (4.31), then $(A_n^*)^\alpha x$ converges weakly to some $w \in H$. Such a w satisfies $\langle w, y\rangle = \langle x, A^\alpha y\rangle$ for every $y \in D(A^\alpha)$, and this implies that $x \in D((A^\alpha)^*) = D((A^*)^\alpha)$ and $w = (A^*)^\alpha x$. Therefore, the domain of A^α is contained in the domain of $(A^*)^\alpha$. Exchanging the roles of A and A^* we get that $D(A^\alpha) = D((A^*)^\alpha)$, and $\lim_{n\to\infty}(A_n^*)^\alpha x = (A^*)^\alpha x$ for each x in the common domain. Letting $n \to \infty$ in (4.31), and in the similar estimate with A^* in the place of A, we get the claimed estimates.

Now we consider the case where $0 \in \sigma(A)$. For each $\varepsilon > 0$ the operator $A + \varepsilon I$ satisfies (4.21) and 0 belongs to its resolvent set, so $D((A+\varepsilon I)^\alpha) = D((A^* + \varepsilon I)^\alpha)$ for $0 \le \alpha < 1/2$, and for every x in the common domain we have

$$\|(A^* + \varepsilon I)^\alpha x\| \le \tan\frac{\pi(1+2\alpha)}{4}\|(A + \varepsilon I)^\alpha x\|,$$

$$\|(A + \varepsilon I)^\alpha x\| \le \tan\frac{\pi(1+2\alpha)}{4}\|(A^* + \varepsilon I)^\alpha x\|.$$

By lemma 4.1.11, $D((A + \varepsilon I)^\alpha) = D(A^\alpha)$ and $(A + \varepsilon I)^\alpha x \to A^\alpha x$ for each $x \in D(A^\alpha)$, and the same holds with A replaced by A^*. Letting $\varepsilon \to 0$ the statement follows. □

Let us consider now the imaginary powers A^{it}, $t \in \mathbb{R}$.

Theorem 4.3.5 *Assume that $A : D(A) \subset H \mapsto H$ is m-accretive and that $0 \in \rho(A)$. Then the imaginary powers A^{it}, $t \in \mathbb{R}$, are bounded operators, and*

$$\|A^{it}\| \le e^{\pi|t|/2}, \;\; t \in \mathbb{R}.$$

Proof. Theorem 4.3.2 implies that the statement is true if A is bounded and satisfies (4.22).

Next step is to show that the statement is true if A is bounded, m-accretive and one to one. This is done considering the operators $A+\varepsilon I$ with $\varepsilon > 0$, which satisfy (4.22) with $\delta = \varepsilon$, and letting $\varepsilon \to 0$. Indeed, since $(A+\varepsilon I)x \to x$ and $(\xi I + A + \varepsilon I)^{-1}x \to (A+\varepsilon I)^{-1}x$ for each $x \in H$, and $(A+\varepsilon I)^{-1}x \to A^{-1}x$ for each $x \in R(A)$ as $\varepsilon \to 0$, we may let $\varepsilon \to 0$ in formula (4.11) getting

$$\|A^{it}x\| = \lim_{\varepsilon\to 0} \|(A+\varepsilon I)^{it}x\| \le e^{\pi|t|/2}\|x\|, \;\; t \in \mathbb{R}.$$

Since A^{it} is closed, we may conclude that $D(A^{it}) = H$ and $\|A^{it}\| \le e^{\pi|t|/2}$ for every $t \in \mathbb{R}$.

The fact that $(A+\varepsilon I)^{-1}x \to A^{-1}x$ for each $x \in R(A)$ as $\varepsilon \to 0$ may be proved as follows: from the equality

$$A((A+\varepsilon I)^{-1}y - A^{-1}y) + \varepsilon((A+\varepsilon I)^{-1}y - A^{-1}y) = -\varepsilon y,$$

which holds for each $y \in H$, we obtain

$$\|A((A+\varepsilon I)^{-1}y - A^{-1}y)\|^2 + \varepsilon^2\|(A+\varepsilon I)^{-1}y - A^{-1}y\| \le \varepsilon^2\|y\|^2,$$

so that $0 = \lim_{\varepsilon\to 0} A((A+\varepsilon I)^{-1}y - A^{-1}y) = \lim_{\varepsilon\to 0}(A+\varepsilon I)^{-1}Ay - A^{-1}Ay$, i.e. $\lim_{\varepsilon\to 0}(A+\varepsilon I)^{-1}x = A^{-1}x$ for every $x = Ay$ in the range of A.

In the final step we consider a general m-accretive operator A with $0 \in \rho(A)$. Therefore A^{-1} is bounded, m-accretive and one to one. It is not hard to see that

$$A^{it} = (A^{-1})^{-it}, \;\; t \in \mathbb{R}.$$

But we already know that $(A^{-1})^{-it}$ is bounded, with norm not exceeding $e^{\pi|t|/2}$. The statement follows. □

Theorems 4.3.5 and 4.2.6 yield the next corollary.

Corollary 4.3.6 *If $A : D(A) \subset H \mapsto H$ is m-accretive and $0 \in \rho(A)$ then for $0 \le \text{Re}\,\alpha < \text{Re}\,\beta$ we have*

$$[D(A^\alpha), D(A^\beta)]_\theta = D(A^{(1-\theta)\alpha+\theta\beta}).$$

4.3.1 Self-adjoint operators in Hilbert spaces

Here H is again a complex Hilbert space, and $A : D(A) \subset H \mapsto H$ is a positive definite self-adjoint operator. A self-adjoint operator is said to be positive if there is $\delta > 0$ such that

$$\langle Ax, x\rangle \ge \delta\|x\|^2, \;\; x \in D(A). \qquad (4.32)$$

Lemma 4.3.7 *If A is a self-adjoint operator satifying (4.32), then $\rho(A)$ contains $(-\infty, \delta)$ and*

$$\|R(\lambda, A)\| \le \frac{1}{\delta - \lambda}, \;\; \lambda < \delta. \qquad (4.33)$$

Proof. Let $\lambda < \delta$, $x \in D(A)$. Then

$$\|(\lambda I - A)x\|^2 = \|(\lambda - \delta)x + (\delta I - A)x\|^2$$

$$= (\lambda - \delta)^2\|x\|^2 + 2(\lambda - \delta)\langle x, \delta x - Ax\rangle + \|(\delta I - A)x\|^2,$$

so that

$$\|(\lambda I - A)x\|^2 \geq (\lambda - \delta)^2\|x\|^2, \tag{4.34}$$

and therefore $\lambda I - A$ is one to one. We prove that it is also onto, showing that its range is both closed and dense in H. Let $x_n \in D(A)$ be such that $\lambda x_n - Ax_n$ converges. From the inequality (4.34) we get

$$\|(\lambda I - A)(x_n - x_m)\|^2 \geq (\lambda - \delta)^2\|x_n - x_m\|^2, \;\; n, m \in \mathbb{N},$$

so that x_n is a Cauchy sequence, λx_n is a Cauchy sequence, and Ax_n is a Cauchy sequence. Then x_n and Ax_n converge; let x, y be their limits. Since A is self-adjoint, then it is closed (we recall that each adjoint operator is closed), so that $x \in D(A)$, $Ax = y$, and $\lambda x_n - Ax_n$ converges to $\lambda x - Ax \in$ Range $(\lambda I - A)$. Therefore, the range of $\lambda I - A$ is closed.

Let y be orthogonal to the range of $(\lambda I - A)$. Then for each $x \in D(A)$ we have $\langle y, \lambda x - Ax\rangle = 0$, so that $y \in D(A^*) = D(A)$ and $\overline{\lambda} y - A^*y = \lambda y - Ay = 0$. Since $\lambda I - A$ is one to one, it follows $y = 0$. Therefore, the range of $(\lambda I - A)$ is dense.

Let us estimate $\|R(\lambda, A)\|$. For $x \in H$, let $u = R(\lambda, A)x$. From the equality $\langle x, u\rangle = \langle \lambda u - Au, u\rangle$ it follows $\langle x, u\rangle \leq (\lambda - \delta)\|u\|^2 \leq 0$, so that

$$(\delta - \lambda)\|u\|^2 \leq |\langle x, u\rangle| \leq \|x\|\,\|u\|$$

which implies $\|R(\lambda, A)\| \leq (\delta - \lambda)^{-1}$. □

In this case it is possible to define the powers A^z for $z \in \mathbb{C}$ through the spectral decomposition of A.

Theorem 4.3.8 *Let $A : D(A) \subset H$ be a self-adjoint operator. There exists a unique family of projections $E_\lambda \in L(H)$, $\lambda \in \mathbb{R}$, with the following properties:*

(i) $E_\lambda E_\mu = E_\mu E_\lambda = E_\lambda \leq E_\mu$, *for* $\lambda < \mu$,

(ii) $\lim_{t\to 0^+} E_{\lambda+t}x = E_\lambda x$, $\lambda \in \mathbb{R}$, $x \in H$,

(iii) $\lim_{\lambda\to -\infty} E_\lambda x = 0$, $\lim_{\lambda\to +\infty} E_\lambda x = x$, $x \in H$,

such that

$$D(A) = \{x \in H : \int_{-\infty}^{+\infty} \lambda^2 d\|E_\lambda x\|^2 < \infty, \;\; Ax = \int_{-\infty}^{+\infty} \mu dE_\mu x, \;\; x \in D(A).$$

The above integrals are meant as improper integrals of Stieltjes integrals (note that $\lambda \mapsto \|E_\lambda x\|^2 = \langle E_\lambda x, x\rangle$ is nonnegative and nondecreasing). The family $\{E_\lambda : \lambda \in \mathbb{R}\}$ is called the spectral resolution or spectral decomposition of A. See e.g. [33, Ch. VIII, sect. 120–121].

If $f : \mathbb{R} \mapsto \mathbb{C}$ is continuous, the operator $f(A)$ is defined by

$$D(f(A)) = \{x \in H : \int_{-\infty}^{+\infty} |f(\lambda)|^2 d\|E_\lambda x\|^2 < \infty\},$$

$$\langle f(A)x, y\rangle = \int_{-\infty}^{+\infty} \mu^k d\langle E_\mu x, y\rangle, \;\; x \in D(f(A)), \; y \in H.$$

It is possible to show (see [33, Ch. IX, sect. 126–128]) that for every $\lambda \in \rho(A)$ we have

$$\langle R(\lambda, A)x, y\rangle = \int_{-infty}^{+\infty} \frac{1}{\lambda - \mu} d\langle E_\mu x, y\rangle, \;\; y \in H,$$

and that this definition agrees with the usual definition in the case where f is a power with integer exponent, i.e.

$$D(A^k) = \{x \in H : \int_{-\infty}^{+\infty} \lambda^{2k} d\|E_\lambda x\|^2 < \infty, \;\; k \in \mathbb{Z},$$

$$\langle A^k x, y\rangle = \int_{-\infty}^{+\infty} \mu^k d\langle E_\mu x, y\rangle, \;\; k \in \mathbb{Z}, \; x \in D(A^k), \; y \in H.$$

More generally, this definition agrees with the definition of the powers A^z for any $z \in \mathbb{C}$.

Theorem 4.3.9 *For every $z \in \mathbb{C}$ we have*

$$D(A^z) = \{x \in H : \; \|A^z x\|^2 = \int_0^{+\infty} \lambda^{2\text{Re} z} d\|E_\lambda x\|^2 < \infty\}, \tag{4.35}$$

and

$$A^z x = \int_0^{+\infty} \lambda^z dE_\lambda x, \;\; x \in D(A^z). \tag{4.36}$$

Proof. Let Re $z < 0$. Then

$$A^z = \frac{1}{2\pi i} \int_{\gamma_{r,\theta}} \lambda^z R(\lambda, A) d\lambda$$

$$\frac{1}{2\pi i} \int_{\gamma_{r,\theta}} \lambda^z \int_0^\infty \frac{1}{\lambda - \mu} \, dE_\mu \, d\lambda = \int_0^\infty \frac{1}{2\pi i} \int_{\gamma_{r,\theta}} \frac{\lambda^z}{\lambda - \mu} \, d\lambda \, dE_\mu$$

$$= \int_0^\infty \mu^z dE_\mu,$$

and the statement holds for Re $z < 0$. If Re $z \geq 0$ let $n >$ Re z, $n \in \mathbb{N}$. By definition, $x \in D(A^z)$ if and only if

$$A^{z-n} x = \int_0^{+\infty} \lambda^{z-n} dE_\lambda x \in D(A^n).$$

For each $y \in H$, $y \in D(A^n)$ iff $\int_0^{+\infty} \mu^{2n} d\|E_\mu y\|^2 < \infty$. Therefore, $x \in D(A^z)$ iff

$$\int_0^{+\infty} \mu^{2n} d\|E_\mu A^{z-n} x\|^2 = \int_0^{+\infty} \mu^{2n} d\|E_\mu \int_0^\infty \lambda^{z-n} dE_\lambda x\|^2$$

$$\int_0^{+\infty} \mu^{2n} d\| \int_0^\mu \lambda^{z-n} E_\lambda x\|^2 = \int_0^{+\infty} \mu^{2\text{Re}\, z} d\|E_\mu x\|^2 < \infty$$

and in this case

$$\langle A^z x, y\rangle = \int_0^{+\infty} \mu^n d_\mu \int_0^{+\infty} \lambda^{z-n} d\langle E_\lambda x, E_\mu y\rangle$$

$$= \int_0^{+\infty} \mu^n d_\mu \int_0^{\mu} \lambda^{z-n} d_\lambda \langle E_\lambda x, y\rangle = \int_0^{+\infty} \mu^z d_\mu \langle E_\mu x, y\rangle.$$

□

The function E_λ is constant on each interval contained in the resolvent set of A. In the case of a positive operator the spectrum of A is contained in $(0,\infty)$, so that all the above integrals are in fact integrals over $(0,\infty)$.

It follows from the definition that if f is bounded, then $f(A)$ is a bounded operator, with norm not exceeding $\|f\|_\infty$. If A satisfies (4.32), then $\sigma(A) \subset [\delta, +\infty)$, so that defining $f(\lambda) = \lambda^{it}$ for $\lambda \geq \delta$ and extending f to a continuous bounded function to the whole $\mathbb{R}$, we obtain that A^{it} is a bounded operator, and $\|A^{it}\| \leq 1$, for every real t.

Example 4.3.10 Let Ω be a bounded open set with C^2 boundary, and let $H = L^2(\Omega)$, $A : D(A) = H^2(\Omega) \cap H^1_0(\Omega)$, $Au = -\Delta u$. It is well known that $\sigma(A)$ consists of a sequence of positive eigenvalues, each of them with finite dimensional eigenspace, and there exists an orthonormal basis of H consisting of eigenfunctions of A. Denoting such a basis by $\{e_n\}_{n\in\mathbb{N}}$ and denoting by λ_n the eigenvalue with eigenfunction e_n, we have

$$E_\lambda u = \sum_{n:\, \lambda_n \leq \lambda} \langle u, e_n\rangle e_n.$$

For every α with positive real part, the domain of A^α consists of those $u \in L^2(\Omega)$ such that

$$\sum_{n=1}^{\infty} \lambda_n^{2\mathrm{Re}\,\alpha} |u_n|^2 < \infty,$$

where $u_n = \langle u, e_n\rangle$, and

$$A^\alpha u = \sum_{n=1}^{\infty} \lambda_n^\alpha u_n e_n, \quad u \in D(A^\alpha).$$

Moreover, $\{e_n/\sqrt{\lambda_n} : n \in \mathbb{N}\}$ is an orthonormal basis of $H^1_0(\Omega)$, with respect to the scalar product $\langle Du, Dv\rangle = \int_\Omega u(x)\overline{v(x)}dx$ (see e.g. [8, §IX.8]).

In the particular case $\Omega = (0,\pi)$ it is easy to see that

$$e_n(x) = \sqrt{2/\pi}\,\sin(nx), \quad \lambda_n = n^2.$$

Therefore, $u \in D(A) = H^2(0,\pi) \cap H^1_0(0,\pi)$ iff $\sum_{n=1}^\infty n^4 |u_n|^2 < \infty$, and $Au = \sum_{n=1}^\infty n^2 u_n e_n$. Taking $\alpha = 1/2$, we get $u \in D(A^{1/2})$ iff $\sum_{n=1}^\infty n^2|u_n|^2 < \infty$, that is iff $u \in H^1_0(0,\pi)$, and $A^{1/2}u = \sum_{n=1}^\infty n u_n e_n$.

Let us come back to the general theory. If A satisfies (4.32), (4.35)-(4.36) could be taken as a definition of A^z, and all the properties of A^z could be deduced, without invoking any result of the previous sections. For instance, it follows immediately that for every $x \in D(A^\alpha)$ the function $z \to A^z x$ is holomorphic in the halfplane Re $z <$ Re α, that $D(A^z)$ depends only on Re z, and if Re $z_1 <$ Re z_2 then $D(A^{z_1}) \supset D(A^{z_2})$.

Most important, it is clear from (4.35) that for every $t \in \mathbb{R}$, A^{it} is a bounded operator, and $\|A^{it}\| \leq 1$. Thanks to theorem 4.2.6, this implies that $[D(A^\alpha), D(A^\beta)]_\theta = D(A^{(1-\theta)\alpha+\theta\beta})$ for $0 \leq \text{Re }\alpha < \text{Re }\beta$. Moreover, these spaces coincide also with the real interpolation spaces $(D(A^\alpha), D(A^\beta))_{\theta,2}$, as the next theorem 4.3.12 shows. For its proof we need a lemma.

Lemma 4.3.11 *Let $L : D(L) \subset H$ be a self-adjoint positive operator. Then iL generates a strongly continuous group of contractions e^{itL} in H.*

Proof. Let us prove that the resolvent set of iL contains $\mathbb{R} \setminus \{0\}$.

Since L is self-adjoint, then $\langle Lx, x\rangle \in \mathbb{R}$ for each $x \in D(L)$, so that Re $\langle iLx, x\rangle = 0$ for each $x \in D(L) = D(iL)$. This implies that for every $\lambda > 0$, $\lambda I - iL$ and $-\lambda I - iL$ are one to one, and $\|(\lambda I - iL)^{-1}x\| \leq \|x\|/\lambda$, for each $x \in (\lambda I - iL)(H)$, $\|(\lambda I + iL)^{-1}x\| \leq \|x\|/\lambda$, for each $x \in (\lambda I + iL)(H)$. Indeed for $\lambda > 0$, $x \in X$, setting $y = R(\lambda, iL)x$ we have

$$\lambda\|y\|^2 \leq \text{Re}\,\lambda\|y\|^2 - \text{Re}\,\langle iLy, y\rangle = \text{Re}\,\langle x, y\rangle \leq \|x\|\,\|y\|$$

so that $\|y\| = \|R(\lambda, iL)x\| \leq \|x\|/\lambda$.

To prove that $\lambda I - iL$ and $-\lambda I - iL$ are onto it is enough to check that their ranges are dense in H. Let y be orthogonal to the range of $\lambda I - iL$, then $\langle \lambda x - iLx, y\rangle = 0$ for each $x \in D(L)$, so that

$$y \in D(\lambda I - (iL)^\star), \;\; \lambda y - (iL)^\star y = \lambda y + iLy = 0,$$

so that $y = 0$. It follows that the range of $\lambda I - iL$ is dense in H. Similarly one shows that the range of $\lambda I + iL$ is dense in H. Consequently, $\mathbb{R} \setminus \{0\} \subset \rho(iL)$ and $\|R(\lambda, iL)\| \leq 1/\lambda$, and this implies the statement. □

Theorem 4.3.12 *Let α, $\beta \in \mathbb{C}$ with Re $\alpha \geq 0$, Re $\beta \geq 0$. Then for every $\theta \in (0, 1)$*

$$[D(A^\alpha), D(A^\beta)]_\theta = (D(A^\alpha), D(A^\beta))_{\theta,2} = D(A^{(1-\theta)\alpha+\theta\beta}).$$

Proof. It is sufficient to prove that for real $\beta > 0$

$$(H, D(A^\beta))_{\theta,2} = D(A^{\beta\theta})$$

and the general statement will follow by interpolation and reiteration.

Since A^β is in its turn a positive self-adjoint operator, iA^β generates a strongly continuous group of operators

$$T(t) = e^{itA^\beta}, \;\; t \in \mathbb{R}.$$

By proposition 3.2.1 $(H, D(A^\beta))_{\theta,2}$ consists of the elements x such that $t \mapsto \psi(t) = t^{-\theta}\|T(t)x - x\| \in L^2_*(0, \infty)$. We have

$$\|\psi\|^2_{L^2_*(0,\infty)} = \int_0^\infty t^{-2\theta}\|T(t)x - x\|^2\frac{dt}{t}$$

$$= \int_0^\infty t^{-2\theta}\int_0^\infty |e^{it\lambda^\beta} - 1|^2 d\|E_\lambda x\|^2\frac{dt}{t} = \int_0^\infty \left(\int_0^\infty \frac{|e^{it\lambda^\beta} - 1|^2}{t^{2\theta}}\frac{dt}{t}\right)d\|E_\lambda x\|^2$$

$$= \int_0^\infty \frac{|e^{i\tau} - 1|^2}{\tau^{2\theta+1}}d\tau \int_0^\infty \lambda^{2\theta\beta}d\|E_\lambda x\|^2 = C\|A^\beta x\|^2,$$

so that $(H, D(A^\beta))_{\theta,2} = D(A^{\beta\theta})$, with equivalence of the norms. □

An important corollary about interpolation in Hilbert spaces follows.

Corollary 4.3.13 *Let H_1, H_2 be Hilbert spaces, with $H_2 \subset H_1$, H_2 dense in H_1. Then for every $\theta \in (0,1)$,*

$$[H_1, H_2]_\theta = (H_1, H_2)_{\theta,2}.$$

Proof. It is known (see e.g. [33, Ch. VIII, sect. 124]) that there exists a self-adjoint positive operator A in H_1 such that $D(A) = H_2$. The statement is now a consequence of theorem 4.3.12. □

Chapter 5

Analytic semigroups and interpolation

Throughout the chapter X is a complex Banach space, and $A : D(A) \subset X \mapsto X$ is a sectorial operator, that is there are constants $\omega \in \mathbb{R}$, $\beta \in (\pi/2, \pi)$, $M > 0$ such that

$$\begin{cases} (i) \quad \rho(A) \supset S_{\beta,\omega} = \{\lambda \in \mathbb{C} : \lambda \neq \omega, |\arg(\lambda - \omega)| < \beta\}, \\ \\ (ii) \quad \|R(\lambda, A)\|_{L(X)} \leq \dfrac{M}{|\lambda - \omega|} \quad \forall \lambda \in S_{\beta,\omega}. \end{cases} \tag{5.1}$$

(5.1) allows us to define a semigroup e^{tA} in X, by means of the Dunford integral

$$e^{tA} = \frac{1}{2\pi i} \int_{\omega+\gamma_{r,\eta}} e^{t\lambda} R(\lambda, A) d\lambda, \quad t > 0,$$

where $r > 0$, $\eta \in (\pi/2, \beta)$, and $\gamma_{r,\eta}$ is the curve $\{\lambda \in \mathbb{C} : |\arg\lambda| = \eta, \ |\lambda| \geq r\} \cup \{\lambda \in \mathbb{C} : |\arg\lambda| \leq \eta, \ |\lambda| = r\}$, oriented counterclockwise. We also set $e^{0A} = I$.

It is possible to show that the function $t \mapsto e^{tA}$ is analytic in $(0, +\infty)$ with values in $L(X)$ (in fact, with values in $L(X, D(A^m))$ for every m), so that e^{tA} is called analytic semigroup generated by A. One sees easily that for $x \in X$ there exists $\lim_{t \to 0} e^{tA}x$ if and only if $x \in \overline{D(A)}$ (and in this case the limit is x). Therefore e^{tA} is strongly continuous if and only if $D(A)$ is dense in X. In any case $e^{tA} \in L(X, D(A^m))$ for every $t > 0$ and $m \in \mathbb{N}$, and $d^m/dt^m e^{tA} = A^m e^{tA}$ for $t > 0$. Moreover there are constants $M_k > 0$ such that

$$\begin{cases} (a) \quad \|e^{tA}\|_{L(X)} \leq M_0 e^{\omega t}, \quad t > 0, \\ \\ (b) \quad \|t^k (A - \omega I)^k e^{tA}\|_{L(X)} \leq M_k e^{\omega t}, \quad t > 0. \end{cases} \tag{5.2}$$

Note that for every $x \in X$ and $0 < s < t$, the function $\sigma \mapsto e^{\sigma A}x$ is in $C^1([s, t]; X) \cap C([s, t]; D(A))$, so that

$$e^{tA}x - e^{sA}x = \int_s^t A e^{\sigma A} x d\sigma = A \int_s^t e^{\sigma A} x d\sigma.$$

The same is true also for $s = 0$, in the sense specified by the following lemma.

Lemma 5.0.14 *For every $x \in X$ and $t \geq 0$, the integral $\int_0^t e^{sA}xds$ belongs to $D(A)$, and*

$$A \int_0^t e^{sA} x ds = e^{tA}x - x.$$

If in addition the function $s \mapsto Ae^{sA}x$ *belongs to* $L^1(0,t;X)$*, then*

$$e^{tA}x - x = \int_0^t Ae^{sA}x ds.$$

Other properties of sectorial operators and analytic semigroups may be found in [32, Ch. 2].

5.1 Characterization of real interpolation spaces

Throughout the section we denote by M_0, M_1 two constants such that $\|e^{tA}\|_{L(X)} \le M_0$, $\|tAe^{tA}\|_{L(X)} \le M_1$, for every $t \in (0,1]$.

Since A is sectorial, the operator $A - \omega I$ satisfies the assumptions of chapter 3, and we may use the results of §3.1 to characterize the interpolation spaces $(X, D(A^m))_{\theta,p}$.

Another characterization, which is very useful in abstract parabolic problems, was found by Butzer and Berens (see e.g. [12]) for $m = 1$.

Proposition 5.1.1 *For* $0 < \theta < 1$, $1 \le p \le \infty$ *we have*

$$(X, D(A))_{\theta,p} = \{x \in X : \varphi(t) = t^{1-\theta}\|Ae^{tA}x\| \in L^p_*(0,1)\},$$

and the norms $\|\cdot\|_{(X,D(A))_{\theta,p}}$ *and*

$$x \mapsto \|x\| + \|\varphi\|_{L^p_*(0,1)}$$

are equivalent.

Proof. For every $x \in (X, D(A))_{\theta,p}$ let $a \in X$, $b \in D(A)$ be such that $x = a + b$. Then

$$t^{1-\theta}\|Ae^{tA}x\| \le t^{1-\theta}\|Ae^{tA}a\| + t^{1-\theta}\|Ae^{tA}b\|$$

$$\le t^{-\theta}M_1\|a\| + t^{1-\theta}M_0\|Ab\| \le \max\{M_0, M_1\}t^{-\theta}(\|a\| + t\|b\|_{D(A)})$$

so that

$$t^{1-\theta}\|Ae^{tA}x\| \le \max\{M_0, M_1\}t^{-\theta}K(t, x, X, D(A)),$$

which implies that $\varphi \in L^p_*(0,1)$ with norm not exceeding $\{M_0, M_1\}\|x\|_{(X,D(A))_{\theta,p}}$ max $\{M_0, M_1\}$.

Conversely, if $\varphi \in L^p_*(0,1)$ write x as

$$x = (x - e^{tA}x) + e^{tA}x = -\int_0^t Ae^{sA}x\, ds + e^{tA}x, \quad 0 < t < 1.$$

Since $t^{1-\theta}\|Ae^{tA}x\| \in L^p_*(0,1)$ the same is true for $t^{1-\theta}v(t)$, where v is the mean value $v(t) = t^{-1}\int_0^t \|Ae^{sA}x\|ds$, thanks to corollary A.3.1, and

$$\|t \mapsto t^{1-\theta}v(t)\|_{L^p_*(0,1)} = \left\|t \mapsto t^{-\theta}\int_0^t \|Ae^{sA}x\|ds\right\|_{L^p_*(0,1)} \le \frac{1}{\theta}\|\varphi\|_{L^p_*(0,1)}.$$

Moreover,

$$\|e^{tA}x\|_{D(A)} = \|e^{tA}x\| + \|Ae^{tA}x\| \le M_0\|x\| + t^{-1+\theta}\varphi(t).$$

Therefore

$$t^{-\theta}K(t,x,X,D(A)) \le t^{-\theta}\int_0^t \|Ae^{sA}x\|ds + t^{1-\theta}M_0\|x\| + \varphi(t),$$

so that $t^{-\theta}K(t,x,X,D(A)) \in L^p_*(0,1)$, with norm not exceeding $C(\|x\| + \|\varphi\|_{L^p_*(0,1)})$. We recall that it belongs to $L^p_*(1,\infty)$, with norm not exceeding $C\|x\|$. Therefore, $x \in (X,D(A))_{\theta,p}$, and the statement follows. □

Proposition 5.1.1 has the following generalization.

Proposition 5.1.2 *For $0<\theta<1$, $1\le p\le\infty$ we have*

$$(X,D(A^m))_{\theta,p} = \{x\in X:\ \varphi_m(t) = t^{m(1-\theta)}\|A^m e^{tA}x\| \in L^p_*(0,1)\},$$

and the norms $\|\cdot\|_{(X,D(A^m))_{\theta,p}}$ *and*

$$x \mapsto \|x\| + \|\varphi_m\|_{L^p_*(0,1)}$$

are equivalent.

Proof. (Sketch) Let $m=2$. The embedding $\subset$ may be proved as in proposition 5.1.1, splitting

$$t^{m(1-\theta)}A^m e^{tA}x = t^{m(1-\theta)}A^m e^{tA}a + t^{m(1-\theta)}A^m e^{tA}b$$

if $x=a+b$. Also the idea of the proof of the other embedding is similar, but now instead of the kernel $Ae^{sA}x$ we have to use $sA^2e^{sA}x$: we have

$$\int_0^t sA^2e^{sA}x\,ds = \int_0^t s\frac{d}{ds}Ae^{sA}x\,ds$$

$$= -\int_0^t Ae^{sA}x\,ds + tAe^{tA}x = x - e^{tA}x + tAe^{tA}x.$$

For every $t\in(0,1)$ we split x as

$$x = \int_0^t sA^2e^{sA}x\,ds + e^{tA}x - tAe^{tA}x.$$

Since $t^{1-2\theta}\|tA^2e^{tA}x\| \in L^p_*(0,1)$ the same is true for $t^{1-2\theta}v(t)$, where v is the mean value $v(t) = t^{-1}\int_0^t \|sA^2e^{sA}x\|ds$. Therefore

$$t \mapsto g(t) = t^{-2\theta}\int_0^t \|sA^2e^{sA}x\|ds \in L^p_*(0,1).$$

So,

$$t^{-2\theta}K(t^2,x,X,D(A^2)) \le t^{-2\theta}\left(\int_0^t \|sA^2e^{sA}x\|ds + t^2\|e^{tA}x - tAe^{tA}x\|_{D(A^2)}\right)$$

$$\le g(t) + t^{2-2\theta}((M_0+M_1)\|x\| + \|A^2e^{tA}x\| + 2M_1\|A^2e^{tA/2}x\|)$$

and $t^{-2\theta}K(t^2,x,X,D(A^2)) \in L^p_*(0,1)$, with norm less than $C(\|\varphi_2\|_{L^p_*(0,1)} + \|x\|)$. Then $t^{-\theta}K(t,x,X,D(A^2)) \in L^p_*(0,1)$, again with norm less than $C'(\|\varphi_2\|_{L^p_*(0,1)} + \|x\|)$, and the statement follows for $m=2$.

If m is arbitrary the procedure is similar, with the kernel $s^{m-1}A^me^{sA}x$ replacing $sA^2e^{sA}x$. □

In the case where θm is not integer, proposition 5.1.2 may be deduced from proposition 3.1.8 and proposition 5.1.1.

5.2 Generation of analytic semigroups by interpolation

In this section we shall use interpolation to check that certain operators are sectorial in suitable functional spaces.

Theorem 5.2.1 *Let Y be any interpolation space between X and $D(A)$. Then the part of A in Y, that is the operator*

$$A_Y : D(A_Y) \mapsto Y, \;\; D(A_Y) = \{y \in D(A) : Ay \in Y\}, \;\; A_Y y = Ay$$

is sectorial in Y.

In particular, for every $\theta \in (0,1)$, $1 \le p \le \infty$. the parts of A in $D_A(\theta,p)$, in $D_A(\theta)$, and in $[X, D(A)]_\theta$ are sectorial operators. Similarly, for every $k \in \mathbb{N}$ the part of A in $D_A(\theta + k, p)$ is sectorial in $D_A(\theta + k, p)$.

Proof. Let $\lambda \in S_{\beta,\omega}$. Since $R(\lambda, A)$ commutes with A on $D(A)$, then $\|R(\lambda, A)\|_{L(D(A))} \le M/|\lambda-\omega|$. By interpolation it follows that $R(\lambda, A) \in L(Y)$ and $\|R(\lambda, A)\|_{L(Y)} \le M/|\lambda-\omega|$, and the statement follows. □

In the next theorem we use the Stein interpolation theorem to prove generation of analytic semigroups in L^p spaces.

Theorem 5.2.2 *Let (Ω, μ) be a σ-finite measure space, and let $T(t) : L^2(\Omega) + L^\infty(\Omega) \to L^2(\Omega) + L^\infty(\Omega)$ be a semigroup such that its restriction to $L^2(\Omega)$ is a bounded analytic semigroup in $L^2(\Omega)$ and its restriction to $L^\infty(\Omega)$ is a bounded semigroup in $L^\infty(\Omega)$. Then the restriction of $T(t)$ to $L^p(\Omega)$ is a bounded analytic semigroup in $L^p(\Omega)$, for every $p \in (2, \infty)$.*

Proof. Let $\theta_0 \in (0, \pi/2)$ and $M > 0$ be such that $T(t)$ has an analytic extension to the sector $\Sigma_2 = \{z \in \mathbb{C} : z \ne 0, \; |arg\ z| < \theta_0\}$, and $\|T(z)\|_{L(L^2)} \le M$ for every $z \in \Sigma_2$, $\|T(t)\|_{L(L^\infty)} \le M$ for every $t > 0$.

Let S be the strip $\{z \in \mathbb{C} : \text{Re}\ z \in [0,1]\}$. For every $r > 0$ and $\theta \in (-\theta_0, \theta_0)$ define a function $h : S \mapsto \Sigma_2$ by

$$h(z) = re^{i\theta(1-z)}, \;\; z \in S,$$

and define a family of operators $\Theta_z \in L(L^2)$ by

$$\Theta_z = T(h(z)), \;\; z \in S.$$

Then $z \mapsto \Theta_z$ is continuous and bounded in S, holomorphic in the interior of S, with values in $L(L^2)$. Consequently, if a is a simple function on Ω and b is a simple function on Λ, the function

$$z \mapsto \int_\Lambda (\Theta_z a)(x) b(x) \nu(dx)$$

is continuous and bounded in S, holomorphic in the interior of S. If $\text{Re}\, z = 1$, $h(z) = re^{\theta\, \text{Im}\, z}$ is a positive real number, so that $\Theta_z \in L(L^\infty)$. Moreover

$$\sup_{t\in\mathbb{R}} \|\Theta_{it}\|_{L(L^2)} \le M, \;\; \sup_{t\in\mathbb{R}} \|\Theta_{1+it}\|_{L(L^\infty)} \le M.$$

By the Stein interpolation theorem 2.1.15, applied with $p_0 = q_0 = 2$, $p_1 = q_1 = +\infty$, for every $s \in (0,1)$ the operator Θ_s has an extension in $L(L^p)$ with $p = 2/(1-s)$, and

$\|\Theta_s\|_{L(L^p)} \le M$. Note that since s runs in $(0,1)$, then p runs in $(2,\infty)$. Θ_s is nothing but $T(re^{i\theta(1-s)})$. Since r and θ are arbitrary, for every $z \neq 0$ with $|\arg z| < \theta_0(1-s) = 2\theta_0/p$ $T(z) \in L(L^p)$, with norm less or equal to M.

Let us show that $z \mapsto T(z)$ is holomorphic in the sector $\Sigma_p = \{z \neq 0 : |\arg z| < 2\theta_0/p\}$ with values in $L(L^p)$ (for the moment we know only that it is bounded).

Let $f \in L^p$, $g \in L^{p'}$, and let $f_n \in L^p \cap L^2$, $g_n \in L^{p'} \cap L^2$, be such that $f_n \to f$ in L^p, $g_n \to g$ in $L^{p'}$. Then the functions $z \mapsto \langle T(z)f_n, g_n\rangle$ are holomorphic in $\Sigma_2 \supset \Sigma_p$, and converge to $\langle T(z)f, g\rangle$ uniformly in Σ_p because $\|T(z)\|_{L(L^p)}$ is bounded in Σ_p. Therefore $z \mapsto \langle T(z)f, g\rangle$ is holomorphic in Σ_p. This implies that $z \mapsto T(z)$ is holomorphic with values in $L(L^p)$. □

Theorem 5.2.2 may be easily generalized as follows: if a semigroup $T(t)$ is analytic in L^{p_0} and bounded in L^{p_1} then it is analytic in L^p for every p in the interval with endpoints p_0 and p_1. But the most common situation is $p_0 = 2$, $p_1 = \infty$. For instance, if

$$\mathcal{A}u = \sum_{i,j=1}^{n} D_i(a_{ij}(x)D_j u)(x), \quad x \in \mathbb{R}^n,$$

where the coefficients a_{ij} are in $H^1_{loc}(\mathbb{R}^n)$ and

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \ge 0, \quad x,\, \xi \in \mathbb{R}^n,$$

then the realization of $\mathcal{A}$ in $L^2(\mathbb{R}^n)$ generates an analytic bounded semigroup in $L^2(\mathbb{R}^n)$, whose restriction to $L^\infty(\mathbb{R}^n)$ is a bounded semigroup in $L^\infty(\mathbb{R}^n)$. The proof may be found in the book of Davies [17, Ex. 3.2.11].

5.3 Regularity in abstract parabolic equations

Throughout the section we fix $T > 0$ and we set

$$M_k = \sup_{0<t\le T+1} \|t^k A^k e^{tA}\|_{L(X)}, \quad k \in \mathbb{N} \cup \{0\}, \tag{5.3}$$

and, for $\alpha \in (0,1)$,

$$M_{k,\alpha} = \sup_{0<t\le T+1} \|t^{k-\alpha} A^k e^{tA}\|_{L(D_A(\alpha,\infty),X)}, \quad k \in \mathbb{N}. \tag{5.4}$$

Let $f : [0,T] \mapsto X$, and $u_0 \in X$. Cauchy problems of the type

$$\begin{cases} u'(t) = Au(t) + f(t), \quad 0 < t < T, \\ \\ u(0) = u_0 \end{cases} \tag{5.5}$$

are called abstract parabolic problems, since the most known examples of sectorial operators are the realizations of elliptic differential operators in the usual functional Banach spaces (L^p spaces, spaces of continuous or Hölder continuous functions in $\mathbb{R}^n$ or in open sets of $\mathbb{R}^n$, etc.).

In this section we will see some optimal regularity results for (5.5) involving the interpolation spaces $D_A(\theta,p)$. To begin with, we need some notation and general results about abstract parabolic problems.

Definition 5.3.1 *Let $T > 0$, let $f : [0,T] \mapsto X$ be a continuous function, and let $u_0 \in X$. Then:*

(i) A function $u \in C^1([0,T];X) \cap C([0,T];D(A))$ is said to be a strict solution *of (5.5) in the interval $[0,T]$ if $u'(t) = Au(t) + f(t)$ for each $t \in [0,T]$, and $u(0) = u_0$.*

(ii) A function $u \in C^1((0,T];X) \cap C((0,T];D(A)) \cap C([0,T];X)$ is said to be a classical solution *of (5.5) in the interval $[0,T]$ if $u'(t) = Au(t) + f(t)$ for each $t \in (0,T]$, and $u(0) = u_0$.*

From definition 5.3.1 it follows easily that if problem (5.5) has a strict solution then

$$u_0 \in D(A), \;\; Au_0 + f(0) \in \overline{D(A)}, \tag{5.6}$$

and if problem (5.5) has a classical solution, then

$$u_0 \in \overline{D(A)}. \tag{5.7}$$

Moreover, any strict solution is also classical.

Proposition 5.3.2 *Let $f \in C([0,T],X)$, and let $u_0 \in \overline{D(A)}$. If u is a classical solution of (5.5), then it is given by the variation of constants formula*

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A} f(s)ds, \;\; 0 \le t \le T. \tag{5.8}$$

Proof. Let u be a classical solution of (5.5) in $[0,T]$, and let $t \in (0,T]$. Since $u \in C^1((0,T];X) \cap C([0,T];X) \cap C((0,T];D(A))$, then $u(t)$ belongs to $\overline{D(A)}$ for $0 \le t \le T$, the function

$$v(s) = e^{(t-s)A}u(s), \;\; 0 \le s \le t,$$

belongs to $C([0,t];X) \cap C^1((0,t),X)$, and

$$v(0) = e^{tA}u_0, \;\; v(t) = u(t),$$

$$v'(s) = -Ae^{(t-s)A}u(s) + e^{(t-s)A}u'(s) = e^{(t-s)A}f(s), \;\; 0 < s < t.$$

Then, for $0 < 2\varepsilon < t$,

$$v(t-\varepsilon) - v(\varepsilon) = \int_\varepsilon^{t-\varepsilon} e^{(t-s)A} f(s)ds,$$

so that letting $\varepsilon \to 0$ we get

$$v(t) - v(0) = \int_0^t e^{(t-s)A} f(s)ds,$$

and the statement follows. □

Proposition 5.3.2 implies that the classical solution of (5.5) is unique. Therefore the strict solution is unique. Unfortunately, in general the function defined by (5.8) is not a classical or a strict solution of (5.5). It is called *mild solution* of (5.5). The first term $t \mapsto e^{tA}u_0$ is OK: it is a classical solution of $w' = Aw$, $w(0) = u_0$ if $u_0 \in \overline{D(A)}$, it is a strict solution if $u_0 \in D(A)$, $Au_0 \in \overline{D(A)}$. The difficulties come from the second term,

$$v(t) = \int_0^t e^{(t-s)A} f(s)ds, \;\; 0 \le t \le T.$$

Its regularity properties are stated in the next proposition.

Proposition 5.3.3 *Let $f \in L^\infty(0,T;X)$. Then, for every $\alpha \in (0,1)$, $v \in C^\alpha([0,T];\, X) \cap C([0,T]; D_A(\alpha,1))$. More precisely, it belongs to $C^{1-\alpha}([0,T];\, D_A(\alpha,1))$, and there is C independent of f such that*

$$\|v\|_{C^{1-\alpha}([0,T];D_A(\alpha,1))} \leq C\|f\|_{L^\infty(0,T;X)}. \tag{5.9}$$

Proof. Due to estimate (5.3), with $k = 0$, v satisfies

$$\|v(t)\| \leq M_0 \int_0^t \|f(s)\|ds \leq M_0 T\|f\|_\infty, \;\; 0 \leq t \leq T. \tag{5.10}$$

Since $\|e^{tA}\|_{L(X)}$ and $\|te^{tA}\|_{L(X,D(A))}$ are bounded in $(0,T]$, by interpolation there is $K_{0,\alpha} > 0$ such that $\|e^{tA}\|_{L(X,D_A(\alpha,1))} \leq K_{0,\alpha}t^{-\alpha}$ for $0 < t \leq T$. Similarly, since $\|tAe^{tA}\|_{L(X)}$ and $\|t^2Ae^{tA}\|_{L(X,D(A))}$ are bounded in $(0,T]$, by interpolation there is $K_{1,\alpha} > 0$ such that $\|Ae^{tA}\|_{L(X,D_A(\alpha,1))} \leq K_{1,\alpha}t^{-\alpha-1}$ for $0 < t \leq T$.

Therefore, $s \mapsto \|e^{(t-s)A}\|_{L(X,D_A(\alpha,1))}$ belongs to $L^1(0,t)$ for every $t \in (0,T]$. Then $v(t)$ belongs to $D_A(\alpha,1)$ for every $\alpha \in (0,1)$, and

$$\|v(t)\|_{D_A(\alpha,1)} \leq K_{0,\alpha}(1-\alpha)^{-1}T^{1-\alpha}\|f\|_{L^\infty(0,T;X)}, \tag{5.11}$$

Moreover, for $0 \leq s \leq t \leq T$,

$$\begin{aligned} v(t) - v(s) &= \int_0^s \left(e^{(t-\sigma)A} - e^{(s-\sigma)A}\right) f(\sigma)d\sigma + \int_s^t e^{(t-\sigma)A}f(\sigma)d\sigma \\ &= \int_0^s d\sigma \int_{s-\sigma}^{t-\sigma} Ae^{\tau A}f(\sigma)d\tau + \int_s^t e^{(t-\sigma)A}f(\sigma)d\sigma, \end{aligned}$$

which implies

$$\begin{aligned} &\|v(t) - v(s)\|_{D_A(\alpha,1)} \leq K_{1,\alpha}\int_0^s d\sigma \int_{s-\sigma}^{t-\sigma} \tau^{-1-\alpha}d\tau\, \|f\|_\infty \\ &+K_{0,\alpha}\int_s^t (t-\sigma)^{-\alpha}d\sigma\, \|f\|_\infty \leq \left(\frac{K_{1,\alpha}}{\alpha(1-\alpha)} + \frac{K_{0,\alpha}}{1-\alpha}\right)(t-s)^{1-\alpha}\, \|f\|_\infty, \end{aligned} \tag{5.12}$$

so that v is $(1-\alpha)$-Hölder continuous with values in $D_A(\alpha,1)$. Estimate (5.9) follows now from (5.11) and (5.12). □

The estimates for $v(t)$ blow up as $\alpha \to 1$. In fact it is possible to show that in general v is not Lipschitz continuous with values in X, nor bounded with values in $D(A)$.

Next lemma is useful because it cuts half of the job in showing that a mild solution is classical or strict.

Lemma 5.3.4 *Let $f \in C([0,T];X)$, let $u_0 \in \overline{D(A)}$, and let u be the mild solution of (5.5). The following conditions are equivalent.*

(a) $u \in C((0,T];D(A))$,

(b) $u \in C^1((0,T];X)$,

(c) u is a classical solution of (5.5).

Moreover the following conditions are equivalent.

(a′) $u \in C([0,T]; D(A))$,

(b′) $u \in C^1([0,T]; X)$,

(c′) u is a strict solution of (5.5).

Proof. Of course, (c) is stronger than (a) and (b). Let us show that if either (a) or (b) holds, then u is a classical solution. Since $u_0 \in \overline{D(A)}$, then $t \mapsto e^{tA}u_0$ belongs to $C([0,T]; X)$. We know already that u belongs to $C([0,T]; X)$. Moreover the integral $\int_0^t u(s)ds$ belongs to $D(A)$, and

$$u(t) = u_0 + A\int_0^t u(s)ds + \int_0^t f(s)ds, \quad 0 \le t \le T. \tag{5.13}$$

Indeed, for every $t \in [0,T]$ we have

$$\begin{aligned}\int_0^t u(s)ds &= \int_0^t e^{sA}u_0 ds + \int_0^t ds \int_0^s e^{(s-\sigma)A} f(\sigma)d\sigma \\ &= \int_0^t e^{sA}u_0 ds + \int_0^t d\sigma \int_\sigma^t e^{(s-\sigma)A} f(\sigma)ds.\end{aligned}$$

By lemma 5.0.14, the integral $\int_\sigma^t e^{(s-\sigma)A} f(\sigma)ds = \int_0^{t-\sigma} e^{\tau A} f(\sigma)d\tau$ is in $D(A)$, and

$$A\int_\sigma^t e^{(s-\sigma)A} f(\sigma)ds = (e^{(t-\sigma)A} - I)f(\sigma) \in L^1(0,t).$$

Therefore the integral $\int_0^t u(s)ds$ belongs to $D(A)$, and

$$A\int_0^t u(s)ds = e^{tA}u_0 - u_0 + \int_0^t (e^{(t-\sigma)A} - 1)f(\sigma)d\sigma, \quad 0 \le t \le T,$$

so that (5.13) holds.

From (5.13) we infer that for every t, h such that t, $t+h \in (0,T]$,

$$\frac{u(t+h) - u(t)}{h} = \frac{1}{h}A\int_t^{t+h} u(s)ds + \frac{1}{h}\int_t^{t+h} f(s)ds. \tag{5.14}$$

Since f is continuous at t, then

$$\lim_{h\to 0} \frac{1}{h}\int_t^{t+h} f(s)ds = f(t). \tag{5.15}$$

Let (a) hold. Then Au is continuous at t, so that

$$\lim_{h\to 0} \frac{1}{h}A\int_t^{t+h} u(s)ds = \lim_{h\to 0} \frac{1}{h}\int_t^{t+h} Au(s)ds = Au(t).$$

By (5.14) and (5.15) we get now that u is differentiable at the point t, with $u'(t) = Au(t) + f(t)$. Since both Au and f are continuous in $(0,T]$, then u' too is continuous, and u is a classical solution.

Let now (b) hold. Since u is continuous at t, then

$$\lim_{h\to 0} \frac{1}{h}\int_t^{t+h} u(s)ds = u(t).$$

On the other hand, by (5.14) and (5.15), there exists the limit

$$\lim_{h\to 0} A\left(\frac{1}{h}\int_t^{t+h} u(s)ds\right) = u'(t) - f(t).$$

Since A is a closed operator, then $u(t)$ belongs to $D(A)$, and $Au(t) = u'(t) - f(t)$. Since both u' and f are continuous in $(0,T]$, then also Au is continuous in $(0,T]$, so that u is a classical solution.

The equivalence of (a'), (b'), (c') may be proved in the same way. □

Now we are ready to prove regularity results for (5.5).

Let u be the mild solution of (5.5), and set $u = u_1 + u_2$, where

$$\begin{cases} u_1(t) = \displaystyle\int_0^t e^{(t-s)A}(f(s) - f(t))ds, \quad 0 \le t \le T, \\ \\ u_2(t) = e^{tA}u_0 + \displaystyle\int_0^t e^{(t-s)A} f(t)ds, \quad 0 \le t \le T. \end{cases} \tag{5.16}$$

Theorem 5.3.5 *Let $0 < \alpha < 1$, $f \in C^\alpha([0,T],X)$, $u_0 \in X$. Then the mild solution u of (5.5) belongs to $C^\alpha([\varepsilon,T],D(A)) \cap C^{1+\alpha}([\varepsilon,T],X)$ for every $\varepsilon \in (0,T)$, and*

(i) if $u_0 \in \overline{D(A)}$, then u is a classical solution of (5.5);

(ii) if $u_0 \in D(A)$ and $Au_0 + f(0) \in \overline{D(A)}$, then u is a strict solution of (5.5), and there is C such that

$$\|u\|_{C^1([0,T],X)} + \|u\|_{C([0,T],D(A))} \le C(\|f\|_{C^\alpha([0,T],X)} + \|u_0\|_{D(A)}); \tag{5.17}$$

(iii) if $u_0 \in D(A)$ and $Au_0 + f(0) \in D_A(\alpha,\infty)$, then u' and Au belong to $C^\alpha([0,T],X)$, u' belongs to $B([0,T];D_A(\alpha,\infty))$, and there is C such that

$$\begin{aligned} &\|u\|_{C^{1+\alpha}([0,T],X)} + \|Au\|_{C^\alpha([0,T],X)} + \|u'\|_{B([0,T],D_A(\alpha,\infty))} \\ &\le C(\|f\|_{C^\alpha([0,T],X)} + \|u_0\|_{D(A)} + \|Au_0 + f(0)\|_{D_A(\alpha,\infty)}). \end{aligned} \tag{5.18}$$

Proof. Thanks to lemma 5.3.4, to prove statements (i) and (ii) it is sufficient to show that u belongs to $C((0,T];D(A))$ in the case where $u_0 \in \overline{D(A)}$, and to $C([0,T];D(A))$ in the case where $u_0 \in D(A)$ and $Au_0 + f(0) \in \overline{D(A)}$. We know already by proposition 5.3.3 that $u \in C^\alpha([\varepsilon,T];X)$ for every $\varepsilon \in (0,T)$, and that $u \in C([0,T];X)$ if $u_0 \in \overline{D(A)}$. So we have to study Au.

Let u_1 and u_2 be defined by (5.16). Then $u_1(t) \in D(A)$ for $t \geq 0$, $u_2(t) \in D(A)$ for $t > 0$, and

$$\begin{cases} (i) \quad Au_1(t) = \int_0^t Ae^{(t-s)A}(f(s) - f(t))ds, \;\; 0 \leq t \leq T, \\ \\ (ii) \quad Au_2(t) = Ae^{tA}u_0 + (e^{tA} - 1)f(t), \;\; 0 < t \leq T. \end{cases} \tag{5.19}$$

If $u_0 \in D(A)$, then (5.19)(ii) holds also for $t = 0$.

Let us show that Au_1 is Hölder continuous in $[0, T]$. For $0 \leq s \leq t \leq T$

$$\begin{aligned} Au_1(t) - Au_1(s) &= \int_0^s A\left(e^{(t-\sigma)A} - e^{(s-\sigma)A}\right)(f(\sigma) - f(s))d\sigma \\ &+(e^{tA} - e^{(t-s)A})(f(s) - f(t)) + \int_s^t Ae^{(t-\sigma)A}(f(\sigma) - f(t))d\sigma, \end{aligned} \tag{5.20}$$

so that, since $A(e^{(t-\sigma)A} - e^{(s-\sigma)A}) = \int_{s-\sigma}^{t-\sigma} A^2 e^{\tau A} d\tau$,

$$\begin{aligned} \|Au_1(t) - Au_1(s)\| &\leq M_2 \int_0^s (s-\sigma)^\alpha \int_{s-\sigma}^{t-\sigma} \tau^{-2} d\tau \, d\sigma \, [f]_{C^\alpha} \\ &+2M_0(t-s)^\alpha [f]_{C^\alpha} + M_1 \int_s^t (t-\sigma)^{\alpha-1} d\sigma \, [f]_{C^\alpha} \\ &\leq M_2 \int_0^s d\sigma \int_{s-\sigma}^{t-\sigma} \tau^{\alpha-2} d\tau \, [f]_{C^\alpha} + (2M_0 + M_1\alpha^{-1})(t-s)^\alpha [f]_{C^\alpha} \\ &\leq \left(\frac{M_2}{\alpha(1-\alpha)} + 2M_0 + \frac{M_1}{\alpha}\right)(t-s)^\alpha [f]_{C^\alpha}. \end{aligned} \tag{5.21}$$

Therefore, Au_1 is α-Hölder continuous in $[0, T]$. Moreover, Au_2 is obviously α-Hölder continuous in $[\varepsilon, T]$: hence, if $u_0 \in \overline{D(A)}$, then $u \in C([0, T], X)$ and $Au \in C((0, T]; X)$, so that, by Lemma 5.3.4, u is a classical solution of (5.5), and statement (i) is proved.

If $u_0 \in D(A)$ we have

$$Au_2(t) = e^{tA}(Au_0 + f(0)) + e^{tA}(f(t) - f(0)) - f(t), \;\; 0 \leq t \leq T, \tag{5.22}$$

so that if $Au_0 + f(0) \in \overline{D(A)}$ then Au_2 is continuous also at $t = 0$, and statement (ii) follows.

In the case where $Ax + f(0) \in D_A(\alpha, \infty)$, from (5.22) we get, for $0 \leq s \leq t \leq T$,

$$\begin{aligned} &\|Au_2(t) - Au_2(s)\| \leq \|(e^{tA} - e^{sA})(Au_0 + f(0))\| \\ &+\|(e^{tA} - e^{sA})(f(s) - f(0))\| + \|(e^{tA} - 1)(f(t) - f(s))\| \\ &\leq \int_s^t \|Ae^{\sigma A}\|_{L(D_A(\alpha,\infty),X)} d\sigma \, \|Au_0 + f(0)\|_{D_A(\alpha,\infty)} \\ &+s^\alpha \left\| A \int_s^t e^{\sigma A} d\sigma \right\|_{L(X)} [f]_{C^\alpha} + (M_0 + 1)(t-s)^\alpha [f]_{C^\alpha} \\ &\leq \frac{M_{1,\alpha}}{\alpha} \|Au_0 + f(0)\|_{D_A(\alpha,\infty)} (t-s)^\alpha + \left(\frac{M_1}{\alpha} + M_0 + 1\right)(t-s)^\alpha [f]_{C^\alpha}, \end{aligned} \tag{5.23}$$

so that also Au_2 is Hölder continuous, and the estimate

$$\|u\|_{C^{1+\alpha}([0,T];X)} + \|Au\|_{C^\alpha([0,T];X)}$$

$$\leq C(\|f\|_{C^\alpha([0,T],X)} + \|u_0\|_{D(A)} + \|Au_0 + f(0)\|_{D_A(\alpha,\infty)})$$

follows easily.

Let us estimate $\|u'(t)\|_{D_A(\alpha,\infty)}$. For $0 \leq t \leq T$ we have, by (5.19),

$$u'(t) = \int_0^t Ae^{(t-s)A}(f(s) - f(t))ds + e^{tA}(Au_0 + f(0)) + e^{tA}(f(t) - f(0)),$$

so that for $0 < \xi \leq 1$

$$\|\xi^{1-\alpha}Ae^{\xi A}u'(t)\| \leq \left\|\xi^{1-\alpha}\int_0^t A^2e^{(t+\xi-s)A}(f(s) - f(t))ds\right\|$$

$$+\|\xi^{1-\alpha}Ae^{(t+\xi)A}(Au_0 + f(0))\| + \|\xi^{1-\alpha}Ae^{(t+\xi)A}(f(t) - f(0))\|$$

$$\leq M_2\xi^{1-\alpha}\int_0^t (t-s)^\alpha(t+\xi-s)^{-2}ds\,[f]_{C^\alpha} \tag{5.24}$$

$$+M_0[Au_0 + f(0)]_{D_A(\alpha,\infty)} + M_1\xi^{1-\alpha}(t+\xi)^{-1}t^\alpha\,[f]_{C^\alpha}$$

$$\leq M_2\int_0^\infty \sigma^\alpha(\sigma+1)^{-2}d\sigma\,[f]_{C^\alpha} + M_0[Au_0 + f(0)]_{D_A(\alpha,\infty)} + M_1[f]_{C^\alpha}.$$

Therefore, $\|u'(t)\|_{D_A(\alpha,\infty)}$ is bounded in $[0,T]$, and the proof is complete. □

Theorem 5.3.6 *Let $0 < \alpha < 1$, $u_0 \in X$, $f \in C([0,T];X) \cap B([0,T];\, D_A(\alpha,\infty))$, and let u be the mild solution of (5.5). Then $u \in C^1((0,T];X) \cap C((0,T];D(A))$, and $u \in B([\varepsilon,T];D_A(\alpha+1,\infty))$ for every $\varepsilon \in (0,T)$. Moreover, the following statements hold.*

(i) if $u_0 \in \overline{D(A)}$, then u is a classical solution;

(ii) if $u_0 \in D(A)$, $Au_0 \in \overline{D(A)}$, then u is a strict solution;

(iii) if $u_0 \in D_A(\alpha+1,\infty)$, then u' and Au belong to $C([0,T];X) \cap B([0,T];\, D_A(\alpha,\infty))$, Au belongs to $C^\alpha([0,T];X)$, and there is C such that

$$\|u'\|_{B([0,T];D_A(\alpha,\infty))} + \|Au\|_{B([0,T];D_A(\alpha,\infty))} + \|Au\|_{C^\alpha([0,T];X)}$$
$$\leq C(\|f\|_{B([0,T];D_A(\alpha,\infty))} + \|u_0\|_{D_A(\alpha+1,\infty)}). \tag{5.25}$$

Proof. Let us consider the function v. We are going to show that it is the strict solution of

$$v'(t) = Av(t) + f(t), \;\; 0 < t \leq T, \;\; v(0) = 0, \tag{5.26}$$

and moreover v' and Av belong to $B([0,T];D_A(\alpha,\infty))$, $Av \in C^\alpha([0,T];X)$, and there is C such that

$$\begin{aligned}&\|v'\|_{B([0,T];D_A(\alpha,\infty))} + \|Av\|_{B([0,T];D_A(\alpha,\infty))} + \|Av\|_{C^\alpha([0,T];X)}\\ &\leq C\|f\|_{B([0,T];D_A(\alpha,\infty))}.\end{aligned} \tag{5.27}$$

For $0 \leq t \leq T$, $v(t)$ belongs to $D(A)$, and

$$\|Av(t)\| \leq M_{1,\alpha}\int_0^t (t-s)^{\alpha-1}ds\,\|f\|_{B(D_A(\alpha,\infty))} = \frac{T^\alpha M_{1,\alpha}}{\alpha}\|f\|_{B(D_A(\alpha,\infty))}. \tag{5.28}$$

Moreover, for $0 < \xi \leq 1$,

$$\begin{aligned}&\|\xi^{1-\alpha}Ae^{\xi A}Av(t)\| = \xi^{1-\alpha}\left\|\int_0^t A^2e^{(t+\xi-s)A}f(s)ds\right\|\\ &\leq M_{2,\alpha}\xi^{1-\alpha}\int_0^t (t+\xi-s)^{\alpha-2}ds\|f\|_{B([0,T];D_A(\alpha,\infty))} \leq\\ &\frac{M_{2,\alpha}}{1-\alpha}\,\|f\|_{B([0,T];D_A(\alpha,\infty))},\end{aligned} \tag{5.29}$$

so that Av is bounded with values in $D_A(\alpha,\infty)$. Let us show that Av is Hölder continuous with values in X: for $0 \leq s \leq t \leq T$ we have

$$\begin{aligned}&\|Av(t) - Av(s)\| \leq \left\|A\int_0^s \left(e^{(t-\sigma)A} - e^{(s-\sigma)A}\right)f(\sigma)d\sigma\right\|\\ &+\left\|A\int_s^t e^{(t-\sigma)A}f(\sigma)d\sigma\right\| \leq M_{2,\alpha}\int_0^s d\sigma\int_{s-\sigma}^{t-\sigma}\tau^{\alpha-2}d\tau\,\|f\|_{B([0,T];D_A(\alpha,\infty))}\\ &+M_{1,\alpha}\int_s^t (t-\sigma)^{\alpha-1}d\sigma\,\|f\|_{B([0,T];D_A(\alpha,\infty))}\\ &\leq \left(\frac{M_{2,\alpha}}{\alpha(1-\alpha)} + \frac{M_{1,\alpha}}{\alpha}\right)(t-s)^\alpha\|f\|_{B([0,T];D_A(\alpha,\infty))},\end{aligned} \tag{5.30}$$

so that Av is α-Hölder continuous in $[0,T]$. Estimate (5.27) follows now from (5.28), (5.29), (5.30). Moreover, thanks to Lemma 5.3.4, v is a strict solution of (5.26).

Let us consider now the function $t \mapsto e^{tA}u_0$.

If $u_0 \in \overline{D(A)}$, $t \mapsto e^{tA}u_0$ is the classical solution of $w' = Aw$, $t > 0$, $w(0) = u_0$. If $u_0 \in D(A)$ and $Au_0 \in \overline{D(A)}$ it is a strict solution. If $x \in D_A(\alpha+1,\infty)$, it is a strict solution, moreover it belongs to $B([0,T];D_A(\alpha+1,\infty)) \cap C^\alpha([0,T];D(A))$. Summing up, the statement follows. □

5.4 Applications to regularity in parabolic PDE's

Consider the problem

$$\begin{cases} u_t(t,x) = \Delta u(t,x) + f(t,x), & 0 \leq t \leq T,\ x \in \mathbb{R}^n,\\ u(0,x) = u_0(x), & x \in \mathbb{R}^n,\end{cases} \tag{5.31}$$

where f and u_0 are continuous and bounded functions. We read it as an abstract Cauchy problem of the type (5.5) in the space $X = C(\mathbb{R}^n)$ with the sup norm, setting $u(t) = u(t, \cdot)$ and $f(t) = f(t, \cdot)$. A is the realization of the Laplace operator Δ in X. It is the generator of the Gauss-Weierstrass analytic semigroup defined by (3.6). Note that the domain of A,

$$D(A) = \{\varphi \in C(\mathbb{R}^n) : \Delta\varphi \text{ (in the sense of distributions) } \in C(\mathbb{R}^n)\}$$

$$= \{\varphi \in C(\mathbb{R}^n) \cap W^{2,p}_{loc}(\mathbb{R}^n)\, \forall p \geq 1 : \Delta\varphi \in C(\mathbb{R}^n)\}$$

contains properly $C^2(\mathbb{R}^n)$. However, we already know that for $0 < \theta < 1$, $\theta \neq 1/2$

$$D_A(\theta, \infty) = C^{2\theta}(\mathbb{R}^n), \quad D_A(\theta + 1, \infty) = C^{2\theta+2}(\mathbb{R}^n).$$

For every $f : [0, T] \times \mathbb{R}^n \mapsto \mathbb{C}$ set $\widetilde{f}(t) = f(t, \cdot)$, $0 \leq t \leq T$. The following statements are easy to be checked.

(i) $\widetilde{f} : [0, T] \mapsto X$ is continuous iff f is continuous, bounded, and for every $t_0 \in [0, T]$ $\lim_{t \to t_0} \sup_{x \in \mathbb{R}^n} |f(t, x) - f(t_0, x)| = 0$;

(ii) if $0 < \alpha < 1$, then $\widetilde{f} \in C^{\alpha}([0, T]; X)$ if and only if f is continuous and bounded, and moreover $\sup_{s \neq t,\, x \in \mathbb{R}^n} |f(t, x) - f(s, x)|/|t - s|^{\alpha} < \infty$;

(iii) if $0 < \alpha < 1$, $\alpha \neq 1/2$, $\widetilde{f} \in B([0, T]; D_A(\alpha, \infty))$ iff $f(t, \cdot) \in C^{2\alpha}(\mathbb{R}^n)$ for every t, and $\sup_{0 \leq t \leq T} \|f(t, \cdot)\|_{C^{2\alpha}(\mathbb{R}^n)} < \infty$.

So, we may apply theorems 5.3.5 and 5.3.6. Theorem 5.3.5 gives

Theorem 5.4.1 *Let $0 < \alpha < 1$, $\alpha \neq 1/2$, and let $f : [0, T] \times \mathbb{R}^n \mapsto \mathbb{C}$ be continuous, bounded, such that $f(\cdot, x) \in C^{\alpha}([0, T])$ for each $x \in \mathbb{R}^n$ and $\sup_{x \in \mathbb{R}^n} \|f(\cdot, x)\|_{C^{\alpha}([0,T])} < \infty$. Let $u_0 \in D(A)$ be such that $\Delta u_0 + f(0, \cdot) \in C^{2\alpha}(\mathbb{R}^n)$. Then problem (5.31) has a unique solution u such that u, u_t, Δu are continuous, bounded, and $\sup_{x \in \mathbb{R}^n} \|u_t(\cdot, x)\|_{C^{\alpha}([0,T])} < \infty$, $\sup_{x \in \mathbb{R}^n} \|\Delta u(\cdot, x)\|_{C^{\alpha}([0,T])} < \infty$, $\sup_{t \in [0,T] \in \mathbb{R}^n} \|u_t(t, \cdot)\|_{C^{2\alpha}(\mathbb{R}^n)} < \infty$.*

Applying theorem 5.3.6 gives

Theorem 5.4.2 *Let $0 < \alpha < 1$, $\alpha \neq 1/2$, and let $f : [0, T] \times \mathbb{R}^n \mapsto \mathbb{C}$ be uniformly continuous, bounded, and such that $\sup_{t \in [0,T]} \|f(t, \cdot)\|_{C^{2\alpha}(\mathbb{R}^n)} < \infty$. Let $u_0 \in C^{2\alpha+2}(\mathbb{R}^n)$. Then problem (5.31) has a unique solution u such that u, u_t, Δu are uniformly continuous, bounded, and $\sup_{t \in [0,T]} \|u_t(t, \cdot)\|_{C^{2\alpha}(\mathbb{R}^n)} < \infty$, $\sup_{t \in [0,T]} \|D_{ij} u(t, \cdot)\|_{C^{2\alpha}(\mathbb{R}^n)} < \infty$, $\sup_{x \in \mathbb{R}^n} \|\Delta u(\cdot, x)\|_{C^{\alpha}([0,T])} < \infty$.*

Putting together theorems 5.3.5 and 5.3.6 we get the Ladyzhenskaja–Solonnikov–Ural'ceva theorem (see [29] for a completely different proof). For simplicity we consider only the case $0 < \alpha < 1/2$. It is convenient to adopt the usual notation: we denote by $C^{\alpha, 2\alpha}([0, T] \times \mathbb{R}^n)$ the space of the bounded functions f such that

$$\sup_{t \neq s,\, x \neq y} \frac{|f(t, x) - f(s, y)|}{|t - s|^{\alpha} + |x - y|^{2\alpha}} < \infty,$$

and we denote by $C^{1+\alpha, 2+2\alpha}([0, T] \times \mathbb{R}^n)$ the space of the bounded functions f with bounded f_t, $D_{ij} f$, such that f_t, $D_{ij} f$ are in $C^{\alpha, 2\alpha}([0, T] \times \mathbb{R}^n)$. It is possible to see that this implies that the space derivatives $D_i f$ are $(1/2 + \alpha)$-Hölder continuous with respect to t, with Hölder constant independent of x.

Theorem 5.4.3 (Ladyzhenskaja–Solonnikov–Ural'ceva) *Let* $0 < \alpha < 1/2$ *and let* $f \in C^{\alpha,2\alpha}([0,T] \times \mathbb{R}^n)$, $u_0 \in C^{2\alpha+2}(\mathbb{R}^n)$. *Then problem (5.31) has a unique solution* $u \in C^{1+\alpha,2+2\alpha}([0,T] \times \mathbb{R}^n)$.

Proof. Almost all follows from patching together the results of the above two theorems. It remains to show that the space derivatives $D_{ij}u$ are time α-Hölder continuous. To this aim we use the fact that $C^2(\mathbb{R}^n)$ belongs to the class $J_{1-\alpha}$ between $C^{2\alpha}(\mathbb{R}^n)$ and $C^{2+2\alpha}(\mathbb{R}^n)$ (see the exercises of chapter 1). Moreover, since $\|u_t(t,\cdot)\|_{C^{2\alpha}(\mathbb{R}^n)}$ is bounded in $[0,T]$, then $t \mapsto u(t,\cdot)$ is Lipschitz continuous with values in $C^{2\alpha}(\mathbb{R}^n)$, with Lipschitz constant $\sup_{0\leq\sigma\leq T} \|u_t(\sigma,\cdot)\|_{C^{2\alpha}}$. So, for $0 \leq s \leq t \leq T$ we have

$$\|u(t,\cdot) - u(s,\cdot)\|_{C^2} \leq C\|u(t,\cdot) - u(s,\cdot)\|_{C^{2\alpha}}^{\alpha} \|u(t,\cdot) - u(s,\cdot)\|_{C^{2+2\alpha}}^{1-\alpha}$$

$$\leq C((t-s)\sup_{0\leq\sigma\leq T} \|u_t(\sigma,\cdot)\|_{C^{2\alpha}})^{\alpha}(2\sup_{\sigma\in[0,T]} \|u(\sigma,\cdot)\|_{C^{2+2\alpha}})^{1-\alpha}$$

$$\leq C'(t-s)^{\alpha}$$

and the statement follows. □

This procedure works also if the Laplacian is replaced by any uniformly elliptic operator with regular and bounded coefficients. Indeed it is known that the realization A of such an operator in $C(\mathbb{R}^n)$ is sectorial, and that $D_A(\alpha,\infty) = C^{2\alpha}(\mathbb{R}^n)$, $D_A(\alpha+1,\infty) = C^{2\alpha+2}(\mathbb{R}^n)$ for $\alpha \neq 1/2$. But the proofs of this properties are not trivial; they rely on the Stewart's theorem [35] which, in its turn, is based on the Agmon–Douglis–Nirenberg theorem [3]. See [32, Ch. 3].

Appendix A

The Bochner integral

A.1 Integrals over measurable real sets

We recall here the few elements of Bochner integral theory that are used in these notes. Extended treatments, with proofs, may be found in the books [7], [37].

X is any real or complex Banach space. We consider the usual Lebesgue measure in $\mathbb{R}$, and we denote by $\mathcal{M}$ the σ-algebra consisting of all Lebesgue measurable subsets of $\mathbb{R}$. If $A \subset \mathbb{R}$, χ_A denotes the characteristic function of the set A.

Definition A.1.1 *A function $f : \mathbb{R} \mapsto X$ is said to be* simple *if there are $n \in \mathbb{N}$, $x_1, .., x_n \in X$, $A_1, .., A_n \in \mathcal{M}$, with* $\text{meas}\, A_i < \infty$ *and $A_i \cap A_j = \emptyset$ for $i \neq j$, such that*

$$f = \sum_{i=1}^{n} x_i \chi_{A_i}.$$

If $I \in \mathcal{M}$, a function $f : I \mapsto X$ is said to be Bochner measurable *if there is a sequence of simple functions $\{f_n\}$ such that*

$$\lim_{n\to\infty} f_n(t) = f(t) \text{ for almost all } t \in I.$$

It is easy to see that every continuous function is measurable.

If $f = \sum_{i=1}^{n} x_i \chi_{A_i}$ is a simple function we set

$$\int_{\mathbb{R}} f(t)dt = \sum_{i=1}^{n} x_i \ \text{meas} A_i. \tag{A.1}$$

Definition A.1.2 *Let $f : \mathbb{R} \mapsto X$. f is said to be* Bochner integrable *if there is a sequence of simple functions $\{f_n\}$ converging to f almost everywhere, such that*

$$\lim_{n\to\infty} \int_{\mathbb{R}} \|f_n(t) - f(t)\| dt = 0.$$

Then $n \mapsto \int_{\mathbb{R}} f_n(t)dt$ is a Cauchy sequence in X. We set

$$\int_{\mathbb{R}} f(t)dt = \lim_{n\to\infty} \int_{\mathbb{R}} f_n(t)dt. \tag{A.2}$$

Arguing as in the case $X = \mathbb{R}$, one sees that $\int_{\mathbb{R}} f(t)dt$ is independent of the choice of the sequence $\{f_n\}$. If f is defined on a measurable set, the above definition can be extended as follows.

Definition A.1.3 *If $I \in \mathcal{M}$ and $f : I \mapsto X$, f is said to be integrable over I if the extension $\tilde{f}$ defined by*

$$\tilde{f}(t) \begin{cases} = f(t), & \text{if } t \in I \\ = 0, & \text{if } t \notin I \end{cases}$$

is integrable. In this case we set

$$\int_I f(t)dt = \int_{\mathbb{R}} \tilde{f}(t)dt. \tag{A.3}$$

If $I = (a, b)$, with $-\infty \le a \le b \le +\infty$, we set as usual

$$\int_{(a,b)} f(t)dt = \int_a^b f(t)dt; \quad \int_b^a f(t)dt = -\int_a^b f(t)dt$$

A simple criterion for establishing whether a function is integrable is stated in the following proposition.

Proposition A.1.4 *Let $I \in \mathcal{M}$, and let $f : I \mapsto X$. Then f is integrable if and only if f is measurable and $t \mapsto \|f(t)\|$ is Lebesgue integrable on I. Moreover,*

$$\left\| \int_I f(t)dt \right\| \le \int_I \|f(t)\|dt. \tag{A.4}$$

From the definition it follows easily that if Y is another Banach space and $A \in L(X, Y)$, then for every integrable $f : I \mapsto X$ the function $Af : I \mapsto Y$ is integrable, and

$$\int_I Af(t)dt = A \int_I f(t)dt.$$

In particular, if $\varphi \in X'$ then for every integrable $f : I \mapsto X$ the function $t \mapsto \langle f(t), \varphi\rangle$ is integrable, and

$$\langle \int_I f(t)dt, \varphi\rangle = \int_I \langle f(t), \varphi\rangle dt.$$

It follows that for every couple of integrable functions f, g it holds

$$\int_I (\lambda f(t) + \mu g(t))dt = \lambda \int_I f(t) + \mu \int_I g(t)dt, \;\; \forall\, \lambda, \mu \in \mathbb{C},$$

Another important commutativity property is the following one.

Proposition A.1.5 *Let X, Y be Banach spaces, and let $A : D(A) \subset X \mapsto Y$ be a closed operator. Let $I \in \mathcal{M}$, let $f : I \mapsto X$ be an integrable function such that $f(t) \in D(A)$ for almost all $t \in I$, and $Af : I \mapsto Y$ is integrable. Then the integral $\int_I f(t)dt$ belongs to $D(A)$, and*

$$A \int_I f(t)dy = \int_I Af(t)dt.$$

A.2 L^p and Sobolev spaces

On the set of all measurable functions on I we define the equivalence relation

$$f \sim g \Longleftrightarrow f(t) = g(t) \text{ for almost all } t \in I. \tag{A.5}$$

Definition A.2.1 *$L^1(I;X)$ is the set of all equivalence classes of integrable functions $f : I \mapsto X$, with respect to the equivalence relation (A.5).*

Since no confusion will arise, in the sequel we shall identify the equivalence class $[f] \in L^1(I;X)$ with the function f itself. We define a norm on $L^1(I,X)$ by setting

$$\|f\|_{L^1(I;X)} = \int_I \|f(t)\|dt. \tag{A.6}$$

We define now the spaces $L^p(I;X)$ for $p > 1$.

Definition A.2.2 *Let $p \in (1,+\infty]$, and $I \in \mathcal{M}$. $L^p(I;X)$ is the set of all equivalence classes of measurable functions $f : I \mapsto X$,with respect to the equivalence relation (A.5), such that $t \mapsto \|f(t)\|$ belongs to $L^p(I)$.*

$L^p(I;X)$ is endowed with the norm

$$\|f\|_{L^p(I;X)} = \left(\int_I \|f(t)\|^p dt\right)^{1/p}, \text{ if } p < \infty \tag{A.7}$$

$$\|f\|_{L^\infty(I;X)} = \text{ ess sup } \{\|f(t)\| : t \in I\} \tag{A.8}$$

Arguing as in the case $X = \mathbb{R}$, it is not difficult to see that for $1 \le p \le \infty$, the space $L^p(I;X)$ is complete.

In the following, if there is no danger of confusion, we shall write $\|f\|_p$ instead of $\|f\|_{L^p(I;X)}$.

To introduce the Sobolev space $W^{1,p}(a,b;X)$ we need a lemma.

Lemma A.2.3 *Let $p \in [1,\infty)$. Then the operator*

$$L_0 : D(L_0) = C^1([a,b];X) \mapsto L^p(a,b;X), \;\; L_0 f = f'$$

is preclosed in $L^p(a,b;X)$, that is the closure of its graph is the graph of a closed operator.

Definition A.2.4 *Let $L : D(L) \subset L^p(a,b;X)$ be the closure of the operator L_0 defined in Lemma (A.2.3). We set*

$$W^{1,p}(a,b;X) = D(L)$$

and we endow it with the graph norm. For every $f \in W^{1,p}(a,b;X)$, Lf is said to be the strong derivative *of f, and we denote it by f'.*

In other words, $f \in W^{1,p}(a,b;X)$ if and only if there is a sequence $\{f_n\} \subset C^1([a,b];X)$ such that $f_n \to f$ in $L^p(a,b;X)$ and $f_n' \to g$ in $L^p(a,b;X)$, and in this case $g = f'$. Moreover we have

$$\|f\|_{W^{1,p}(a,b;X)} = \|f\|_{L^p(a,b;X)} + \|f'\|_{L^p(a,b;X)} \;\; \forall f \in W^{1,p}(a,b;X).$$

Since L is a closed operator, then $W^{1,p}(a,b;X)$ is a Banach space.

Let $f \in W^{1,p}(a,b;X)$, and let $\{f_n\} \subset C^1([a,b];X)$ be such that $f_n \to f$ and $f_n' \to f'$ in $L^p(a,b;X)$. From the equality

$$f_n(t) - f_n(s) = \int_s^t f_n'(\sigma)d\sigma \tag{A.9}$$

we get, integrating with respect to s in (a,b) and letting $n \to \infty$,

$$f(t) = \frac{1}{b-a}\bigg(\int_a^b f(s)ds + \int_a^t (\sigma - a)f'(\sigma)d\sigma\bigg), \text{ a.e.in } (a,b).$$

Therefore, $W^{1,p}(a,b;X)$ is continuously embedded in $C([a,b];X)$. Letting $n \to \infty$ in (A.9) we get also

$$f(t) - f(s) = \int_s^t f'(\sigma)d\sigma.$$

Sometimes it is easier to deal with weak (or distributional) derivatives, defined as follows.

Definition A.2.5 *Let $f \in L^p(a,b;X)$. A function $g \in L^1(a,b;X)$ is said to be the weak derivative of f in (a,b) if*

$$\int_a^b f(t)\varphi'(t)dt = -\int_a^b g(t)\varphi(t)dt, \ \ \forall \varphi \in C_0^\infty(a,b).$$

It can be shown that weak and strong derivatives do coincide. More precisely, the following proposition holds.

Proposition A.2.6 *Let $f \in W^{1,p}(a,b;X)$. Then f is weakly differentiable, and f' is the weak derivative of f.*

Conversely, if $f \in L^p(a,b;X)$ admits a weak derivative $g \in L^p(a,b;X)$, then $f \in W^{1,p}(a,b;X)$, and $g = f'$.

A.3 Weighted L^p spaces

Let I be an interval contained in $(0,+\infty)$. For $1 \le p \le \infty$ we denote by $L^p_*(I)$ the space of the L^p functions in I with respect to the measure dt/t, endowed with its natural norm

$$\|f\|_{L^p_*(I)} = \bigg(\int_0^{+\infty} |f(t)|^p \frac{dt}{t}\bigg)^{1/p}, \text{ if } p < \infty,$$

$$\|f\|_{L^\infty_*(I)} = \text{ess sup}_{t\in I}\, |f(t)|.$$

Dealing with L^p_* spaces, the Hardy-Young inequalities are often more useful than the Hölder inequality. They hold for every positive measurable function $\varphi : (0,a) \mapsto \mathbb{R}$, $0 < a \le \infty$, and every $\alpha > 0$, $p \ge 1$. See [23, p.245-246].

$$\begin{cases} (i) \quad \displaystyle\int_0^a t^{-\alpha p}\bigg(\int_0^t \varphi(s)\frac{ds}{s}\bigg)^p \frac{dt}{t} \le \frac{1}{\alpha^p}\int_0^a s^{-\alpha p}\varphi(s)^p\frac{ds}{s}, \\ \\ (ii) \quad \displaystyle\int_0^a t^{\alpha p}\bigg(\int_t^a \varphi(s)\frac{ds}{s}\bigg)^p \frac{dt}{t} \le \frac{1}{\alpha^p}\int_0^a s^{\alpha p}\varphi(s)^p\frac{ds}{s} \end{cases} \tag{A.10}$$

The measure $m(dt) = dt/t$ is the Haar measure of the multiplicative group $\mathbb{R}_+$. So, it is invariant under multiplication: $m(A) = m(\lambda A)$, for every measurable $A \subset \mathbb{R}_+$ and $\lambda > 0$, and $\|\varphi\|_{L^p_*(0,\infty)} = \|\varphi(\lambda\cdot)\|_{L^p_*(0,\infty)}$.

For every $\alpha \neq 0$ the space $L^p_*(0,\infty)$ is invariant under the change of variable $t \mapsto t^\alpha$, in the sense that $\varphi \in L^p_*(0,\infty)$ iff $t \mapsto \varphi(t^\alpha) \in L^p_*(0,\infty)$, and

$$\|\varphi\|_{L^p_*(0,\infty)} = |\alpha|^{1/p}\|t \mapsto \varphi(t^\alpha)\|_{L^p_*(0,\infty)}.$$

(This is obviously true also for $p = \infty$, with the usual convention $1/\infty = 0$). In particular, for $\alpha = -1$ we get an isometry:

$$\|\varphi\|_{L^p_*(0,\infty)} = \|t \mapsto \varphi(t^{-1})\|_{L^p_*(0,\infty)}.$$

Moreover, the change of variable $t \mapsto t^{-1}$ is an isometry also between $L^p_*(1,\infty)$ and $L^p_*(0,1)$.

If X is any Banach space and $1 \leq p \leq \infty$ the space $L^p_*(I;X)$ is the set of all Bochner measurable functions $f : I \mapsto X$, such that $t \mapsto \|f(t)\|_X$ is in $L^p_*(I)$. It is endowed with the norm

$$\|f\|_{L^p_*(I;X)} = \|t \mapsto \|f(t)\|_X\|_{L^p_*(I)}.$$

In chapters 1 and 3 we have used the following consequence of inequality (A.10)(i).

Corollary A.3.1 *Let u be a function such that $t \mapsto u_\theta(t) = t^\theta u(t)$ belongs to $L^p_*(0,a;X)$, with $0 < a \leq \infty$, $0 < \theta < 1$ and $1 \leq p \leq \infty$. Then also the mean value*

$$v(t) = \frac{1}{t}\int_0^t u(s)ds, \quad t > 0 \tag{A.11}$$

has the same property, and setting $v_\theta(t) = t^\theta v(t)$ we have

$$\|v_\theta\|_{L^p_*(0,a;X)} \leq \frac{1}{1-\theta}\|u_\theta\|_{L^p_*(0,a;X)} \tag{A.12}$$

Appendix B

Vector-valued holomorphic functions

Let X be a complex Banach space, let Ω be an open subset of $\mathbb{C}$, $f : \Omega \to X$ be a continuous function and $\gamma : [a,b] \to \Omega$ be a piecewise C^1-curve. The integral of f along γ is defined by

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

As usual, we denote by X' the dual space of X consisting of all linear bounded operators from X to $\mathbb{C}$. For each $x \in X$, $x' \in X'$ we set $x'(x) = \langle x, x' \rangle$.

Definition B.0.2 *f is holomorphic in Ω if for each $z_0 \in \Omega$ the limit*

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} := f'(z_0)$$

exists in X. f is weakly holomorphic in Ω if it is continuous in Ω and the complex-valued function $z \mapsto \langle f(z), x' \rangle$ is holomorphic in Ω for every $x' \in X'$.

Clearly, any holomorphic function is weakly holomorphic; actually, the converse is also true, as the following theorem shows.

Theorem B.0.3 *Let $f : \Omega \to X$ be a weakly holomorphic function. Then f is holomorphic.*

Proof. Let $\overline{B(z_0, r)}$ be a closed ball contained in Ω; we prove that for all $z \in B(z_0, r)$ the following Cauchy integral formula holds:

$$f(z) = \frac{1}{2\pi i} \int_{\partial B(z_0,r)} \frac{f(\xi)}{\xi - z} d\xi. \tag{B.1}$$

First of all, we observe that the right hand side of (B.1) is well defined because f is continuous. Since f is weakly holomorphic in Ω, the complex-valued function $z \mapsto \langle f(z), x' \rangle$ is holomorphic in Ω for all $x' \in X'$, and hence the ordinary Cauchy integral formula in $B(z_0, r)$ holds, i.e.,

$$\langle f(z), x' \rangle = \frac{1}{2\pi i} \int_{\partial B(z_0,r)} \frac{\langle f(\xi), x' \rangle}{\xi - z} d\xi = \left\langle \frac{1}{2\pi i} \int_{\partial B(z_0,r)} \frac{f(\xi)}{\xi - z} d\xi, x' \right\rangle.$$

Since $x' \in X'$ is arbitrary, we obtain (B.1). We can differentiatee with respect to z under the integral, so that f is holomorphic and

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial B(z_0,r)} \frac{f(\xi)}{(\xi - z)^{n+1}}\, d\xi$$

for all $z \in B(z_0, r)$ and $n \in \mathbb{N}$. □

Definition B.0.4 *Let $f : \Omega \to X$. We say that f admits a power series expansion around a point $z_0 \in \Omega$ if there exist a X-valued sequence (a_n) and $r > 0$ such that $B(z_0, r) \subset \Omega$ and*

$$f(z) = \sum_{n=0}^{+\infty} a_n (z - z_0)^n \quad \text{in } B(z_0, r).$$

Theorem B.0.5 *Let $f : \Omega \to X$ be a continuous function; then f is holomorphic if and only if f has a power series expansion around every point of Ω.*

Proof. Assume that f is holomorphic in Ω. Then, if $z_0 \in \Omega$ and $B(z_0, r) \subset \Omega$, the Cauchy integral formula (B.1) holds for every $z \in B(z_0, r)$.

Fix $z \in B(z_0, r)$ and observe that the series

$$\sum_{n=0}^{+\infty} \frac{(z - z_0)^n}{(\xi - z_0)^{n+1}} = \frac{1}{\xi - z}$$

converges uniformly for ξ in $\partial B(z_0, r)$, since $\big|(z - z_0)/(\xi - z_0)\big| = r^{-1}|z - z_0|$. Consequently, by (B.1) we obtain

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\partial B(z_0,r)} f(\xi) \sum_{n=0}^{+\infty} \frac{(z - z_0)^n}{(\xi - z_0)^{n+1}}\, d\xi \\ &= \sum_{n=0}^{+\infty} \Big[\frac{1}{2\pi i} \int_{\partial B(z_0,r)} \frac{f(\xi)}{(\xi - z_0)^{n+1}}\, d\xi \Big] (z - z_0)^n, \end{aligned}$$

the series being convergent in X.

Conversely, suppose that

$$f(z) = \sum_{n=0}^{+\infty} a_n (z - z_0)^n, \quad z \in B(z_0, r),$$

where (a_n) is a sequence with values in X. Then f is continuous, and for each $x' \in X'$,

$$\langle f(z), x' \rangle = \sum_{n=0}^{+\infty} \langle a_n, x' \rangle (z - z_0)^n, \quad z \in B(z_0, r).$$

This implies that the complex-valued function $z \mapsto \langle f(z), x' \rangle$ is holomorphic in $B(z_0, r)$ for all $x' \in X'$ and hence f is holomorphic by Theorem B.0.3. □

Now we extend some classical theorems of complex analysis to the case of vector-valued holomorphic functions.

Theorem B.0.6 (Cauchy) *Let $f : \Omega \to X$ be holomorphic in Ω and let D be a regular domain contained in Ω. Then*

$$\int_{\partial D} f(z)dz = 0.$$

Proof. For each $x' \in X'$ the complex-valued function $z \mapsto \langle f(z), x'\rangle$ is holomorphic in Ω and hence

$$0 = \int_{\partial D} \langle f(z), x'\rangle dz = \langle \int_{\partial D} f(z)dz, x'\rangle.$$

□

Remark B.0.7 [improper complex integrals] As in the case of vector-valued functions defined on a real interval, it is possible to define *improper complex integrals* in an obvious way. Let $f : \Omega \to X$ be holomorphic, with $\Omega \subset \mathbb{C}$ possibly unbounded. If $I = (a, b)$ is a (possibly unbounded) interval and $\gamma : I \to \mathbb{C}$ is a piecewise C^1 curve in Ω, then we set

$$\int_\gamma f(z)dz := \lim_{s \to a^+,\, t \to b^-} \int_s^t f(\gamma(\tau))\gamma'(\tau)d\tau,$$

provided that the limit exists in X.

Theorem B.0.8 (Laurent expansion) *Let $f : D := \{z \in \mathbb{C} : r < |z - z_0| < R\} \to X$ be holomorphic. Then, for every $z \in D$*

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z - z_0)^n,$$

where

$$a_n = \frac{1}{2\pi i} \int_{\partial B(z_0, \varrho)} \frac{f(z)}{(z - z_0)^{n+1}}\, dz, \quad n \in \mathbb{Z},$$

and $r < \varrho < R$.

Proof. Since for each $x' \in X'$ the function $z \mapsto \langle f(z), x'\rangle$ is holomorphic in D the usual Laurent expansion holds, that is

$$\langle f(z), x'\rangle = \sum_{n=-\infty}^{+\infty} a_n(x')(z - z_0)^n$$

where the coefficients $a_n(x')$ are given by

$$a_n(x') = \frac{1}{2\pi i} \int_{\partial B(z_0, \varrho)} \frac{\langle f(z), x'\rangle}{(z - z_0)^{n+1}}\, dz, \quad n \in \mathbb{Z}.$$

It follows that

$$a_n(x') = \langle a_n, x'\rangle, \quad n \in \mathbb{Z},$$

where the a_n are those indicated in the statement. □

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